SPECTRAL PROPERTIES OF SOME ERGODIC SYSTEMS

BY

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CONTENTS

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<u>ABSTRACT</u>

This thesis deals with the spectral properties of some dynamical systems.

In Chapter I one of the main tools necessary for this work will be reviewed. This is the spectral theory of unitary operator in Hilbert spaces. Chapter II deals with tensor products of unitary operat (hence direct product of invertible measure preserving transformation, i.m.p.t) In this chapter we develop a technique (the $\mathcal{Q}$-technique) which enable us to compute a multiplicity pair for the tensor products of two or more unitary operators. The chapter ends with an application of the main theorem to operators (i.m.p.t's) with simple discrete spectrum.

The other main tool needed for this work is the theory of Gaussian processes and will be reviewed in Chapter III. Chapter IV deals with some invariant σ-algebras for measure preserving transformation. A generalization $\mathcal{Q}_\theta(T)$ of a canonical σ-algebra $\mathcal{A}_\theta(T)$ defined by Walters [28] will be given. The properties of the σ-algebras $\mathcal{Q}_\theta(T)$ will be studied. The spectral properties of transformation with $\mathcal{A}_\theta = \mathcal{B}$ will be investigated. Also, after Parry [19], we introduce the concept of representations in $\mathcal{C}_\theta$, the class of all transformations with $\mathcal{A}_\theta = \mathcal{B}$. It will be proved that if $\mathcal{A}_\theta(T) = \mathcal{B}$ then the sequence $T_i = T^{\theta(i)}$ converges to a

limit in the group of all transformations on a Lebesgue space. The set of all such limits will be found to form a group G(T), and G(T) is a conjugacy invariant. The algebras $\mathcal{A}_\theta(T)$ will be studied in relation to the concept of mixing and in relation to entropy theory. The relation of these σ-algebras to group extension and Gaussian processes will be considered.

CHAPTER 0

INTRODUCTION

Measure theoretic ergodic theory can be viewed as the study of the category $\mathcal{X}$ of measure spaces as objects and measure preserving transformations as maps. Such a viewpoint would be useful if there was another category (or number of categories), about which more is known, to which $\mathcal{X}$ is functorially related. The category $\mathcal{H}$ of Hilbert spaces as objects and linear operators as maps is an example, an important one, of these categories. The functor F will be the map which sends a measure space to its L^2-space and a measure preserving transformation to its induced linear operator.

The importance of the category $\mathcal{H}$ comes from the fact that the operator in L^2 induced by an invertible measure transformation is unitary; and a unitary operator has a multiplicity pair as a complete set of invariants (i.e. one that classify the unitary operator up to unitary equivalence). By a multiplicity pair we mean a maximal spectral type and a multiplicity function. Thus this set of invariants will be attached to an invertible measure preserving transformation through the above relation. However, this set is not complete for all transformations (i.e. one which classify it up to a conjugacy). A class for which this set of invariants is complete is the class of ergodic measure preserving transformation with discrete spectrum.

The category $\mathcal{X}$ is closed under direct products and the category $\mathcal{H}$ is closed under tensor products. Moreover, the functor F takes direct products in $\mathcal{X}$ to tensor products in $\mathcal{H}$. In Chapter II, we give a computation of multiplicity pair of tensor products of unitary operators. (As far as I know this computation is new). The hope is that it might help in throwing more light on the structure of $\mathcal{X}$. A class of transformation for which that might help is non-ergodic measure preserving transformation with discrete spectrum and for which partial results have been obtained by Choksi [2].

One of the conjugacy invariants for an invertible measure preserving transformation T is the class of T-invariant sub-σ-algebras of $\mathcal{B}$ (T acts on a Lebesgue space $(X,\mathcal{B},m)$. Since Halmos [10] posed the question "what are the possibilities for non-trivial T-invariant sub-σ-algebras", a few such sub-σ-algebras have been found. Pinsker [21] defined and studied the algebras $\Pi(T)$ where $\Pi(T)$ is the maximum partition with zero entropy for T. Using Kushnirenko definition, [13], of sequence entropy, another invariant sub-σ-algebra can be defined in the same way. Walters [28], defined canonically, an invariant sub-σ-algebra $\mathcal{A}'_\theta(T)$ for an invertible measure preserving T and a sequence of integers $\theta = \{\theta(i)\}$. This sub-σ-algebra of sets which stay, a.e. the same, asymptotically under the θ-itertates of T. Formally it is defined as $\mathcal{A}'_\theta(T) = \{A \in \mathcal{B} \mid m(T^{\theta(i)}A \,\Delta\, A) \to 0\}$. In his work [28], Walters studied these sub-σ-algebras in relation to entropy theory, mixing, group extensions and Gaussian processes. The importance of the relation to entropy theory comes from the result: if $\mathcal{A}'_\theta(T) = \mathcal{B}$ then T is in the class of transformations with zero entropy; a class for which a lot of work yet to be done. As for mixing, transformations with $\mathcal{A}'_\theta(T) = \mathcal{B}$ are disjoint from all strong mixing transformation. Group extensions and Gaussian processes, provide some examples with $\mathcal{A}'_\theta(T) = \mathcal{B}$.

In Chapter IV, we take Walters' idea further and define a sub-σ-algebra $\mathcal{A}_\theta(T)$ for an invertible measure preserving transformation T and a sequence of integers θ. $\mathcal{A}_\theta(T)$ could be regarded as the sub-σ-algebras of sets on which, asymptotically the effect of θ-iterates of T discribed with a single invertible measure preserving transformation. Formally $\mathcal{A}_\theta(T) = \{A \in \mathcal{B} \mid m(T^{\theta(i)}A \,\Delta\, T^{\theta(j)}A) \to 0\}$. $\mathcal{A}_\theta(T)$ contains $\mathcal{A}'_\theta(T)$ and retains most of its properties. This will be proved through modifications of Walters' proofs.

It has been proved in [28] that the class $\mathcal{C}'_\theta$ of all transformations have $\mathcal{A}'_\theta = \mathcal{B}$, is closed under reasonable finite operations. The class $\mathcal{C}_\theta$, which contains $\mathcal{C}'_\theta$, will be proved to have the same property. Following Parry's idea [19] of a structure theory based on models, we will introduce the concept of representation in $\mathcal{C}_\theta$. It will be proved also that any ergodic transformation with sufficiently many representations in $\mathcal{C}_\theta$, is metrically isomorphic to an inverse limit of $\mathcal{C}_\theta$ transformations and is itself in $\mathcal{C}_\theta$.

The set $G(T)$ of all invertible measure preserving transformation which can be regarded, asymptotically, as θ-iterates of T for some θ, will be shown to be a commutative group. It will be called the T-group and it will be shown to be a conjugacy invariant.

If T is a group extension, by $Z_2 = \{-1,1\}$, of the form $T(y,z) = (\beta y, \varphi(y)z)$ then transformations in $G(T)$ will be described in terms of transformations in $G(\beta)$ and a function related to φ.

If T_μ is a Gaussian shift based on a covariance measure μ the transformation in $G(T_\mu)$ will be proved to be generalized Gaussian. Also $G(T_\mu)$ will be proved to be homomorphic to another useful group $\mathcal{G}(T_\mu)$ namely the group of $L^2(K,\mu)$-limit points of (λ^n). Let μ be concentrated on $D \cup D^{-1}$, where D is a Kronecker subset of K, then T is weak mixing [17], and there exists a sequence θ such that $\mathcal{Q}'_\theta(T) = \mathcal{B}$ [28]. We shall show, using the fact that continuous functions supported on D are uniformly approximated by powers of λ, that there exist a sequence θ such that $\mathcal{A}_\theta(T) = \mathcal{B}$. Moreover, $G(T_\mu)$, in this case, will be isomorphic to $C(D,K)$ = continuous functions supported on D and of modulus 1. Therefore the powers of T_μ are dense in $G(T_\mu)$. Theorem (15) of [28], will be generalized to give a group of weak mixing Gaussian shifts with $\mathcal{A}_\theta(T) = \mathcal{B}$.

The main tools, that have been employed in this work are the spectral theory of unitary operators and the theory of Gaussian processes. Therefore, we review both of them in Chapter I and Chapter III respectively. The results of those two chapters can be found, in more detail; in [3], [7], [9], [20] and [23].

The main theorem of Chapter II, theorem (6.5), gives a multiplicity pair for the tensor product of two unitary operators. An application of the main theorem to the case of symmetric tensor product of two unitary operators with simple discrete spectrum will be given in §9.

Chapter IV deals with some invariant sub-σ-algebras for a measure preserving transformation. The definition and properties are given in §1. §2 deals with relation of $\mathcal{A}_\theta(T)$ to entropy theory. It is then, that the concept of representation in $\mathcal{C}_\theta$ is presented. The T-group is dealt with in §3 and it is proved there that G(T) is a conjugacy invariant. §4 deals with the sub-σ-algebras $\mathcal{A}_\theta(T)$ and mixing properties. Relations to group extension are given in §5 and relations to Gaussian processes are given in §6.

CHAPTER 1.

SPECTRAL THEORY

The first two sections of this chapter deal with definitions and results from Hilbert space theory and measure theory needed here. The last section introduces a canonical form for a unitary operator in a Hilbert space. The main references of this chapter are [3], [9] and [23].

§1. Hilbert space theory.

Definition 1.1.

Throughout the following F will denote either or , where is the field of real numbers and is the field of complex numbers. $\mathbb{N}$ will denote the set of integers.

Let H be a vector space over F. An inner product or scalar product in a vector space H is an F-valued function $(\, , \,): H \times H \to F$ s.t. (i) $(x,y) = \overline{(y,x)}$ $\forall x,y \in H$,

(ii) $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1(x_1,y) + \alpha_2(x_2,y)$, $\forall x_1,x_2 \in H, y \in H$, $\forall \alpha_1,\alpha_2 \in F$

(iii) $(x,x) \geq 0$; $(x,x) = 0 \iff x = 0$.

A vector space H equipped by an inner product is called a pre-Hilbert space. The quantity $(x,x)^{\frac{1}{2}} = \|x\|$ will be named the norm of x. The norm defines a metric d on H where $d(x,y) = \|x - y\|$.

H with the metric d is a metric space. A Hilbert space H is a complete pre-Hilbert space.

A countable family (x_n) of vectors in a Hilbert space H is said to be generating or spanning if the closure of the linear manifold (the set of all finite linear combinations of members of (x_n)) is the whole space H.

From now on the term Hilbert space means separable Hilbert space.

Definition 1.2.

A linear transformation A from a Hilbert space H_1 to another Hilbert space H_2 is bounded if $\exists\ \alpha \in \mathbb{R}$ s.t.

$$\|Ax\|_{H_2} \leq \alpha \|x\|_{H_1}$$

An operator A in a Hilbert space H is a bounded linear transformation from H into itself. An operator A is said to be invertible if $\exists$ an operator B s.t. AB = BA = I, where I is the identity operator and juxtaposition means composition of two operators. The operator B when it exists will be written A^{-1}. The adjoint A^* of an operator A in a Hilbert space H is defined for any pair $x, y \in H$ from $(Ax,y) = (x,A^*y)$. An operator A is self-adjoint if $A^* = A$. A projection P on a subspace $\mathfrak{M}$ of H is an idempotent, self-adjoint operator.

An operator U in a Hilbert space H is normal if $U U^* = U^* U$. A normal operator is unitary if $U U^* = U^* U = I$. If U is unitary then the following is true: (i) $U^{-1}=U^*$, (ii) $(Ux,Uy) = (x,y), \forall\ x,y \in H$.

Definition 1.3.

The spectrum $\sigma(A)$ of an operator A in a Hilbert space H is the closed set in $\mathbb{C}$ (usual topology) s.t. the operator $(A - \lambda I)$ is not invertible i.e. $\sigma(A) = \{\lambda \in \mathbb{C} \mid (A-\lambda I) \text{ is not invertible}\}$. If A is unitary then $\sigma(A) \subset K = \{Z \in \mathbb{C} \mid |Z| = 1\}$.

Definition 1.4.

A unitary operator U in a Hilbert space H is cyclic if a vector $h \in H$ s.t. the family $(U^n h, n \in N)$ generates the whole space H. In this case H will be called cyclic and h will be called a generator.

Definition 1.5.

Let $(X, \mathcal{B}, m)$ be a measure space. The space $L^2(X, \mathcal{B}, m)$ of all complex valued functions having square integrable modulii is a Hilbert space with inner product

$$(f,g) = \int_X f\, \bar{g}\, d\, m\,, \qquad f, g \quad L^2(X, \mathcal{B}, m).$$

Here only separable L^2 spaces will be considered.

Lemma 1.6.

If m is a measure defined on the Borel sets of the circle $K = \{Z \in \mathbb{C} \mid |Z| = 1\}$ then in the space $L^2(K, \mathcal{B}, m)$, the multiplication operator defined by $M_m : f(\lambda) \to \lambda f(\lambda)$, $\forall f \in L^2(K, \mathcal{B}, m)$ is unitary. Moreover $L^2(K, \mathcal{B}, m)$ is cyclic under M_m with generator $f \equiv 1$.

Definition 1.7.

Two unitary operators U and V in a Hilbert space H are unitarily equivalent if $\exists$ a unitary operator W in H s.t. $WUW^{-1} = V$. If U and V are two unitary operators in two Hilbert spaces H_1 and H_2 then U is unitarily equivalent to V if $\exists$ an invertible isometry $\emptyset$ from H_1 to H_2 s.t. $\emptyset U \emptyset^{-1} = V$. Since we are going to deal only with unitary equivalence it will be abbreviated equivalence.

§2. Measure Theory.

Here, in this section, some facts from measure theory will be presented. Throughout the section the term "measure" will be used for a finite positive measure. Nevertheless the subsequent results hold for more general measures. (See [8]).

Definition 2.1.

Let μ,ν be two measures on the measurable space $(X, \mathcal{B})$. $\mu \ll \nu$ viz μ is absolutely continuous wrt ν iff $\nu(M) = 0 \Rightarrow \mu(M)=0$, $\forall M \in \mathcal{B}$. $\mu \sim \nu$ viz μ is equivalent to ν iff $\mu \ll \nu$ and $\nu \ll \mu$. "$\sim$" is an equivalence relation in the set of measures.

Theorem 2.2.

Radon-NiKodym Theorem:

If μ,ν are two measures s.t. $\nu \ll \mu$ then $\exists$ a non-negative function $f \in L^1(\mu)$ s.t.

$$\nu(M) = \int_M f \, d\mu \quad , \quad \forall M \in \mathcal{B} .$$

, f is unique up to sets of μ-measure zero.

Lemma 2.3.

If μ, ν are two measures such that $\nu << \mu$ then $\exists$ a set N in $\mathcal{B}$ such that $\nu \sim \mu_N$ where μ_N is the measure defined for all $M \in \mathcal{B}$ by $\mu_N(M) = \mu(M \cap N)$, $\forall M \in \mathcal{B}$.

The set N mentioned above is $N = \{x \in X | f(x) > 0\}$ where f is the R - D function defined in R - D theorem (2.2). Such a function f is said to be an R - D derivative of ν wrt μ and is written $\frac{d\nu}{d\mu}$.

§3. The Canonical Form.

The canonical form for a unitary operator in a Hilbert space will be introduced here. Of course, the first two sections will be needed, but, any term, definition or result will be mentioned without reference.

Let $K = \{Z \in \mathbb{C} \mid |z| = 1\}$ and $\mathcal{B}$ be its Borel σ-field.

Definition 3.1.

A spectral type $\bar{x}$ of a vector $x \in H$ relative to a unitary operator U in H is defined from

$$(U^n x, x) = \int_K \lambda^n d\,\bar{x} . \quad \bar{x} \text{ is a positive finite measure on } (K, \mathcal{B}).$$

A spectral type $\bar{x}$ of a vector $x \in H$ is said to be maximal if it dominates all other spectral types, i.e. $\bar{x} >> \bar{y}$, $\forall y \in H$. In a cyclic Hilbert space, a maximal spectral type is one which corresponds to a generator.

Theorem 3.2.

If U is a cyclic unitary operator in a Hilbert space H with x as a generator and $\bar{x}$ is its spectral type, then U is equivalent to the multiplication operator $M_{\bar{x}}$ in $L^2(K, \mathcal{C}, \bar{x})$

Proof

1 is a generator in $L^2(K, \mathcal{C}, \bar{x})$

x is a generator in H.

Let $\emptyset$ be the function taking $U^n x \in H$, $n \in Z$ to $\lambda^n \in L^2(K, \mathcal{C}, \bar{x})$

Clearly $\emptyset$ is an isometry as $(U^n x, U^m x)_H = (U^n U^{-m} x, x) = \int_K \lambda^{n-m} d\bar{x}$

$$= \int_K \lambda^n \bar{\lambda^m} \, d\bar{x} = (\lambda^n, \lambda^m)_{L^2(K, \bar{x})}.$$

Therefore, extension by linearity proves equivalence. Q.E.D.

The above theorem can be extended to more general Hilbert spaces which are not cyclic under unitary operators. It will turn out that, relative to a unitary operator U, H can be made equivalent to the direct sum of L^2-spaces with suitable measures. Furthermore, this can be arranged in a way that enables one to construct what is called a multiplicity function.

Now, the following lemma will be given without proof.

Lemma 3.3.

Let U be a unitary operator in a Hilbert space. Let x_o be a given vector in H. Then there is an $x \in H$, maximal relative to U, s.t. x_o lies in the cyclic subspace $M(x)$ generated by x.

<u>Note</u>. A maximal element is an element corresponding to the maximal spectral type.

<u>Theorem 3.4</u>.

Let U be a unitary operator in H. Then, a sequence $(v_n) \subset H$ s.t. $H = \sum_{i=1}^{\infty} \oplus H(v_i)$, where $H(v_i)$ is the cyclic space generated by v_i.

<u>Proof</u>.

Let (y_n) be an orthonormal basis in H. Using the above lemma, an element v_1 can be selected in such a way that $y_1 \in H(v_1)$ and v_1 is maximal. Let U_2 be the restriction of U to the orthocomplement $H(v_1)^1$ of $H(v_1)$ in H. Since U maps $H(v_1)$ into itself and $H(v_1)^1$ into itself, it follows that U_2 is unitary.

Now V_2 in $H(v_1)^1$ can be selected in such a way that v_2 is maximal relative to U_2, $H(v_2)$ contains the perpendicular projection of y_2 on $H(v_1)^1$. Then y_1 and y_2 are in $H(v_1) \oplus H(v_2)$. And so on the process continues to give

$$H = \sum_{i=1}^{\infty} \oplus H(v_i).$$

<u>Remark.</u>

Since v_1 in the last lemma is maximal, its spectral type dominates all other spectral types. In particular it dominates v_2. Since v_2 is maximal in the orthocomplement of $H(v_1)$; it dominates, as well, all other spectral types. In particular it dominates v_3, and so on. The conclusion is the corresponding sequence (μ_n) of measures with the property $\mu_n >> \mu_{n+1}$, $n \geq 1$.

Now Theorems 3.2, 3.4 and remark lead to the following theorem which gives the canonical form for a unitary operator.

Theorem 3.5.

Let U be a unitary operator in a Hilbert space H. H can be made equivalent to the direct sum $\sum_{n\geq 1} \oplus\ L^2(K,\mu_n)$ s.t.

i) $\mu_n >> \mu_{n+1}$, $n \geq 1$

ii) $U \simeq M_{\mu_1} \oplus M_{\mu_2} \oplus \ldots\ldots.$.

Using R - D theorem and results of §2 and since $\mu_1 >> \mu_n$ then

$$\mu_n(M) = \int_M f_n(\lambda)\, d\mu_1(\lambda),\ M \in \mathcal{G},$$

$\mathcal{G}$ is the Borel sets in K. Let $M_n = \{\lambda \in K,\ f_n(\lambda) > 0\}$. Thus modifying f_{n+1}, if necessary on a set of μ_1-measure zero, it may be supposed that the sequence (M_n) is decreasing and M_1 is the whole circle K.

Definition 3.6.

Let μ be a measure on K and (M_n) be a decreasing sequence of Borel sets in K with M = K. Let U be a unitary operator in a Hilbert space H. If H is equivalent to $\sum_n \oplus\ L^2(M_n,\mu)$ in the sense that U is equivalent to the direct sum of corresponding multiplication operators, then the equivalence will be called ordered-representation. The measure μ is called the measure of it while the sets (M_n) will be called the multiplicity sets. If $\mu(M_k) > 0$ and $(M_{k+1}) = 0$ then the multiplicity of the representation will be K and if $\mu(M_k) > 0$, for all k the system will have infinite multiplicity.

Given the sets M_n in the above definition and the maximal spectral type μ_1 of U in H, $\Sigma \, \chi_{M_n}$ is defined to be the <u>multiplicity function</u> of U. It must be noted that two multiplicity functions are equal a.e. w.r.t.μ

The conclusion is the maximal spectral type of a unitary operator U in a Hilbert space H together with the constructed multiplicity function characterize U in the following sense.

Two unitary operators in a Hilbert space H are unitarily equivalent if they have the same <u>multiplicity pair</u>.

By a <u>multiplicity pair</u> we mean (μ, f) where μ is the maximal spectral type and f is the corresponding multiplicity function.

CHAPTER II

SPECTRAL ANALYSIS AND TENSOR PRODUCTS

The main object of the present chapter is to give a multiplicity pair for the tensor product of two unitary operators in the tensor product of two Hilbert spaces. The idea will be generalized to more than two unitary operators.

Since an invertible measure preserving transformation on a measure space induces a unitary operator in the L^2-space over that measure space we will be able to sum the results of the chapter in a functorial language. The main categories will be the category of Hilbert spaces and the category of measure space.

The chapter will be concluded by an application of the main theorem to obtain a multiplicity pair for the symmetric tensor product of a unitary operator, with simple discrete spectrum, with itself.

The word function in an L^2-space will mean a class of functions i.e. functions which are equal a.e. . A Hilbert space will mean a separable Hilbert space.

§1. Tensor Products

Definition 1.1.

Let H_1 and H_2 be two Hilbert spaces with inner products $<,>_1$, $<\ >_2$. Let $H_1 \times H_2$ be their cartesian product and $H_1 \circ H_2$ be the free linear space over $H_1 \times H_2$. For any two elements

$x_1 = \sum_{j=1}^{n} a_j\,(h_j^1, h_j^2)$ and $x_2 = \sum_{j=1}^{n} b_j\,(h_j^{'1}, h_j^{'2})$, define a form

$$\langle x_1, x_2 \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j\,\bar{b}_k\,\langle h_j^1, h_k^{1'} \rangle_1\,\langle h_j^2, h_k^{2'} \rangle_2 .$$

The set N of all null elements in $H_1 \circ H_2$ i.e. the set of elements $x \in H_1 \circ H_2$ such that $\langle x,x \rangle = 0$ is a subspace of $H_1 \circ H_2$ where the form $<,>$ can be easily seen to be an inner product in $H_1 \circ H_2$. The form $<,>$, also, defines an inner product in the quotient space $H_1 \circ H_2/N$ which will be denoted $<,>$ as well. The completion of $H_1 \circ H_2/N$ with respect to $<,>$ will be denoted by $H_1 \otimes H_2$ and will be called the tensor product of H_1 and H_2. $h_1 \otimes h_2$ will be the image of $(h_1, h_2) \in H_1 \times H_2$ under the map from $H_1 \times H_2$ into $H_1 \otimes H_2$, for all $h_1 \in H_1$, $h_2 \in H_2$.

Proposition 1.2.

Let $(X_i, \mathcal{B}_i, m_i)$, $i = 1,2$ be two measure spaces and $L^2(X_i, \mathcal{B}_i, m_i)$, $i = 1,2$, denotes their L^2-spaces. Then the tensor product $\bigotimes_{i=1}^{2} L^2(X_i, \mathcal{B}_i, m_i) = L^2(X_1, \mathcal{B}_1, m_1) \otimes L^2(X_2, \mathcal{B}_2, m_2)$ is isomorphic to the L^2-space defined on the product measure space $(X_1, \mathcal{B}_1, m_1) \times (X_2, \mathcal{B}_2, m_2) = (X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, m_1 \times m_2)$

Proof. See [16].

Remark

Both the definition and the proposition, could be generalized to more than two spaces in an obvious manner.

Definition 1.3.

Let U and V be unitary operators in the Hilbert spaces H_1 and H_2 respectively. Define $U \otimes V$ in $H_1 \otimes H_2$ by $U \otimes V\ (h_1 \otimes h_2) = Uh_1 \otimes Vh_2$, $h_1 \in H_1$ and $h_2 \in H$. Then it can be easily shown that $U \otimes V$ extends to a unitary operator. Also, this definition carries through to more than two operators. The notation $U^{n\otimes}$ in $H^{n\otimes}$ will stand for $\underbrace{U \otimes \ldots \otimes U}_{n}$ in $\underbrace{H \otimes \ldots \otimes H}_{n}$

§2. Convolution of measures and maximal spectral types.

It is known (cf. CH.1) that the multiplication operator $M_m \in L^2(K,\mathcal{B},m)$ is a cyclic unitary operator. Also, following proposition 1.2, $L^2(K,\mathcal{B},m_1) \otimes L^2(K,\mathcal{B},m_2)$ is isomorphic to $L^2(K \times K, \mathcal{B} \times \mathcal{B}, m_1 \times m_2)$. If M_{m_i}, $i = 1,2$ denote the corresponding multiplication operator in $L^2\ (K,\mathcal{B},m_i)$, $i = 1,2$, then by a natural correspondence $M_{m_1} \otimes M_{m_2}$ could be made equivalent to the extension of the operator which takes $f(z,w) \in L^2(K\times K, \mathcal{B}\times\mathcal{B}, m_1 \times m_2)$ to the $zwf(z,w)$ i.e. the operator of the multiplication by the two arguments. We will speak indifferently of both the operators of multiplication by the two arguments and $M_{m_1} \otimes M_{m_2}$.

Definition 2.1.

The convolution of two measures μ and ν on locally compact abelian group G is defined from the equation

$$\int_G f\,d(\mu*\nu) = \int_G\int_G f(xy)\ d\mu(x)\ d\nu(y),$$

where the group is written multiplicatively, f is a continuous function with compact support and μ, ν are in $M(G)$ ($M(G)$ is the set of all bounded regular complex valued measures on G). For more details we refer to [26]. The convolution concept could be carried out for any number of measures.

Lemma 2.2.

A maximal spectral type $M_{m_1} \otimes M_{m_2}$ in $L^2(K\times K, \mathcal{B}\times\mathcal{B}, m_1 \times m_2)$ is $m_1 * m_2$.

Proof.

This follows at once from the following sequence of relations

$$((M_{m} \otimes M_{m_2})^n 1, 1) = \int_{K \times K} z^n w^n \, d(m_1 \times m_2) = \int_K z^n \, d(m_1 * m_2),$$

and since the spectral types of $z^k w^\ell$, ℓ, k, $\ell \in \mathbb{Z}$ are dominated by $(m_1 * m_2)$.

Lemma 2.3.

A maximal spectral type of $\bigotimes_{i=1}^{n} M_{m_2}$ in $L^2(K^n, \mathcal{B}^n, \mathop{X}_{i=1}^{n} m_1)$, where $K^n = Kx \ldots xK$, n times, $\mathcal{B}^n$ is the product σ-algebra, is $m_1 * \ldots . * m_n = \mathop{*}_{i=1}^{n} m_i$.

Proof.

The proof is the same as the proof of lemma 2.2 with appropriate changes.

§3. Equivalence

In this section, the operator $M_{m_1} \otimes M_{m_2}$ will be proved to be equivalent to the operator W, where W is the operator of multiplication by the second argument. Let $K = \{z \in \mathbb{C} \mid |z| = 1\}$ and $K^2 = K \times K$.

Let ϕ be the point map in K^2 defined by

$$\phi : K^2 \to K^2$$

$$: (z,w) \mapsto (z, z^{-1}w)$$

where z^{-1} is the inverse in the multiplicative group K. Then the inverse map ϕ^{-1} will be

$$\phi^{-1} : K^2 \to K^2$$

$$: (z,w) \to (z, zw).$$

ϕ induces a new measure in $(K^2, m_1 \times m_2)$ which will be denoted by $(m_1 \times m_2) \circ \phi$. Also ϕ induces a map of $L^2(m_1 \times m_2)$ into $L^2(m_1 \times m_2 \circ \phi)$, which will be denoted A_ϕ.

Lemma 3.1.

A_ϕ is an invertible isometry.

Proof.

That A_ϕ is invertible follows from the relation $A^{-1}_\phi = A_{\phi^{-1}}$.

As for the isometric property, it follows from

$$\| A_{\emptyset}^{-1} f \|_{m_1 \times m_2} = \int_{K^2} |f(\emptyset^{-1}(z,w))|^2 \, d\,(m_1 \times m_2)\,(z,w)$$

$$= \int_{K^2} |f(z,w)|^2 \, d((m_1 \times m_2) \circ \emptyset)(z,w)$$

$$= \| f \|_{(m_1 \times m_2) \circ \emptyset}, \text{ for all } f \in L^2(m_1 \times m_2).$$

<u>Lemma 3.2.</u>

The operator $M_{m_1} \otimes M_{m_2}$ in $L^2(K \times K,\ m_1 \times m_2)$ is equivalent to the operator W in $L^2\ (K \times K,\ m_1 \times m_2 \circ \emptyset)$ which is the extension of the map $f(z,w) \to wf(z,w)$, $f \in L^2(K \times K,\ (m_1 \times m_2) \circ \emptyset$.

<u>Proof.</u>

The equivalence will be established through $A_{\emptyset}$ as follows

$A_{\emptyset} \circ (M_{m_1} \otimes M_{m_2}) \circ A_{\emptyset}^{-1} = W$, Since

$A_{\emptyset}^{-1} f(z,w) = f(z,w)$,

$(M_{m_1} \otimes M_{m_2} \circ A_{\emptyset}^{-1}) f(z,w) = zw\ f(z,zw)$

and

$(A_{\emptyset} \circ M_{m_1} \otimes M_{m_2} \circ A_{\emptyset}^{-1} f(z,w) = wf\ (z,w)$.

Lemma 3.1 and 3.2 will be generalized for the case of n-dimensional torus $K^n = K \times \ldots \times K$. Let

$\emptyset_n^{-1} : K^n \to K^n$

$: (z_1, \ldots, z_n) \to (z_1, z_1 z_2\ \ldots, z_{n-1} z_n)$, then $\phi_n : K^n \to K^n$

$$: (z_1, \ldots, z_n) \to (z_1, z_1^{-1} z_2, \ldots, z_{n-1}^{-1} z_n).$$

Also, A_{ϕ_n} will be the induced map of $L^2(K^n, \overset{n}{\underset{i=1}{X}} m_i) \to L^2(K^n, (\overset{n}{\underset{i=1}{X}} m_i) \circ \phi_n)$,

where $A^{-1}_{\phi_n} = A_{\phi_n^{-1}}$.

Lemma 3.3.

A_{ϕ_n} is an invertible isometry.

Proof.

The proof is essentially the same as that of lemma 3.1, with, of course, the suitable changes.

Lemma 3.4.

The operator $\overset{n}{\underset{i=1}{X}} M_{m_i}$ in $L^2(\overset{n}{\underset{i=1}{X}} m_i)$ is equivalent to the operator W_n in $L^2((\overset{n}{\underset{i=1}{X}} m_i) \circ \phi_n)$ defined as the extension of the map which takes $f(z_1, \ldots, z_n) = z_n f(z_1, \ldots, z_n)$, $f \in L^2(\overset{n}{\underset{i=1}{X}} m_i)$

Proof.

See lemma 3.2.

§4. The $\mathcal{Q}$-Technique

Throughout this section $(X, \mathcal{B}, m)$ will denote a normalized measure space i.e. $m(X) = 1$ and $\mathcal{Q}$ is a sub-σ-algebra of $\mathcal{B}$. The definitions, lemmas and theorems are valid for any $\mathcal{Q}$, but it is fixed once it is chosen. $L^2(X, \mathcal{B}, m)$ will denote, as usual, the space of complex valued functions of square integrable modulii.

Definition 4.1.

Let $f: X \to \mathbb{C}$ be a measurable integrable function. If $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}$, then $E(f|\mathcal{A})$ is the essentially unique $\mathcal{A}$ measurable function satisfying

i) it is $\mathcal{A}$-measurable and integrable

ii) $\int_A E(f|\mathcal{A})dm = \int_A f\, dm, \qquad A \in \mathcal{A}$.

$E(f|\mathcal{A})$ is called the conditional expectation of f with respect to $\mathcal{A}$. If f is as above and g is bounded measurable function, then

$$E(gf|\mathcal{A}) = g\, E(f|\mathcal{A}) \quad \text{a.e.} \quad [m].$$

For more details about conditional expectation we refer to [14].

In the following, the conditional expectation concept will be used to define a quantity which will be called an $\mathcal{A}$-norm. It is nothing but the square root of the conditional expectation of the squared modulus of a function in $L^2(X,\mathcal{B},m)$ with respect to $\mathcal{A}$. One might ask why is it called a norm? The answer will be, of course, it is not a norm in the usual sense, for it is a function and not a scalar. However, an $\mathcal{A}$-norm enjoys some properties, and plays some roles, which look like those of an ordinary norm. Take e.g. the concept of normalization, if f is a vector then $f/\|f\|$ has norm 1. Something similar to the concept of normalization will be defined using the $\mathcal{A}$-norm. Namely, it is the $\mathcal{A}$-normalization. The $\mathcal{A}$-normalization will be defined for a vector f, through dividing it by the $\mathcal{A}$-norm of f. But since the $\mathcal{A}$-norm of f is a function this requires some elaboration, while the $\mathcal{A}$-norm of f

$\mathcal{A}$-normalized need not be one, but rather a function namely χ_A where $A \in \mathcal{A}$.

That was about properties. As for the roles, through normalization, the norm plays an essential role in the celebrated Gramm-Schmidt orthogonalization process. Something similar to the aforementioned process, using $\mathcal{A}$-normalization, and which plays an essential role, will be defined. Of course, then both the concept of spanning and orthogonalization will be changed as well.

One final comment in this heurestic account is about the normalization process. The norm $\|f\|$ of $f \in L^2(X, \mathcal{B}, m)$ might be looked at as the square root of the conditional expectation of $|f|^2$ with respect to the trivial σ-algebra $\mathcal{N}$ (i.e. the σ-algebra of sets of measure zero and sets of measure one). So, normalization is in fact $\mathcal{N}$-normalization. Or normalization over a single atom of $\mathcal{N}$ namely X. Following this line of thinking, an $\mathcal{A}$-norm is a sort of normalization over the atoms of the σ-algebra $\mathcal{A}$ or a fibre-normalization!

Definition 4.2.

Let $(X, \mathcal{B}, m)$ and $\mathcal{A}$ as above. Then

i) An $\mathcal{A}$-norm of $f \in L^2(X, \mathcal{B}, m)$ is defined as

$$\|f\|_{\mathcal{A}} = \{E(|f|^2|\mathcal{A})\}^{\frac{1}{2}} \quad \text{a.e.}$$

ii) An $\mathcal{A}$-inner product of $f, g \in L^2(X, \mathcal{B}, m)$ is defined as

$$(f, g)_{\mathcal{A}} = E(f\bar{g}|\mathcal{A}) \quad \text{a.e.}$$

So f is $\mathcal{A}$-orthogonal to g if $(f,g)_{\mathcal{A}} = 0$ a.e.

Lemma 4.3.

Let $f \in L_\infty(X,\mathcal{B},m)$, $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}$. Let $A = \{x \in X: \|f\|_{\mathcal{A}} \neq 0\}$. Define g as

$$g(x) = \frac{f(x)}{\|f\|_{\mathcal{A}}(x)}, \quad x \in A \quad \text{a.e.}$$
$$= 0 \quad x \in A^c, \text{ then}$$
$$\|g\|_{\mathcal{A}} = \chi_A \quad \text{a.e.}$$

Note: The conditional expectation can be defined for positive functions and the formula $E(uv|\mathcal{A})$ holds of $\mathcal{A}$-measurable u, not necessarily bounded.

Proof of lemma

g can be rewritten as

$$g = \frac{\chi_A f}{\{E(|f|^2|\mathcal{A})\}^{\frac{1}{2}}+\chi_{A^c}} \quad \text{a.e., then}$$

$$|g|^2 = \frac{\chi_A |f|^2}{E(|f|^2|\mathcal{A})+\chi_{A^c}} \quad \text{a.e.}$$

By taking the conditional expectation of both sides with respect to $\mathcal{A}$, we have

$$E(|g|^2|\mathcal{A}) = \frac{\chi_A}{E(|f|^2|\mathcal{A})+\chi_{A^c}} \quad E(|f|^2|\mathcal{A}) = \chi_A$$

Thus $\|g\|_{\mathcal{A}} = \chi_A$ a.e.

Definition 4.4.

An $\mathcal{A}$-normalization of $f \in L_\infty(X, \mathcal{B}, m)$ is the function g defined in lemma 4.3.

Note: $g \in L^2$ since $\int_X |g|^2 \, dm = \int_A E(|g|^2 | \mathcal{A}) dm = \int_A \chi_A \, dm < \infty$.

Definition 4.5.

Let X stand for the measure space $(X, \mathcal{B}, m)$. If A is a measurable set, then A will be used as well for the sub-measure space induced on it by restriction.

By $\mathcal{A}$-spanning of L^2 we mean considering L^2 as an $L_\infty^{(\mathcal{A})}$-module.

Definition 4.6.

If we replace, in the Gram-Schmidt orthogonalization process, scalars by L_∞-functions, inner products by $\mathcal{A}$-inner products, orthogonalization by $\mathcal{A}$-orthogonalization, norm by $\mathcal{A}$-norm and spanning by $\mathcal{A}$-spanning we get a new process which will be called $\mathcal{A}$-Gram-Schmidt orthogonalization process.

It is known that $L_\infty(X, \mathcal{B}, m)$ is dense in $L^2(X, \mathcal{B}, m)$ and also there exist a countable family in $L_\infty(X, \mathcal{B}, m)$ which span $L^2(X, \mathcal{B}, m)$.

Lemma 4.7.

Let $\{f_i\}_{i \in I} \subset L_\infty(X, \mathcal{B}, m)$ span $L^2(X, \mathcal{B}, m)$. Then given a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$, a family $\{g_i\}_{i \in I}$ along with a family $\{A_i\}_{i \in I} \subset \mathcal{A}$, can be found using $\mathcal{A}$-Gram Schmidt orthogalization process such that

$$\langle g_i, g_j \rangle_{\mathcal{A}} = 0 \quad \text{a.e.} \quad i \neq j$$

$$\|g_i\|^2_{\mathcal{A}} = \langle g_i, g_i \rangle_{\mathcal{A}} = \chi_{A_i} \quad \text{a.e.} \quad i = j \ , \ i,j \in I.$$

Moreover the family $\{g_i\}_{i \in I}$ $\mathcal{A}$ -span $L^2(X,\mathcal{B},m)$

Proof

The proof will be divided into steps

Step 1

Consider f_1 and the $\mathcal{A}$-normalization g_1 of f_1 then $\|g_1\|_{\mathcal{A}} = \chi_{A_1}$ a.e. $A_1 = \{x \in X : E(|f_1|^2|\mathcal{A}) \neq 0\}$ and $A_1 \in \mathcal{A}$. By the definition of the conditional expectation, the set on which f_1 does not vanish is included in the set A_1. So, since $f_1 = \|f\|_{\mathcal{A}}\, g_1$ we conclude that f_1 is in the $\mathcal{A}$ -span of g_1.

Step 2

If $f \in L_\infty(X,\mathcal{B},m)$, $g \in L^2(X,\mathcal{B},m)$ and $E(|g|^2|\mathcal{A}) = \chi_A$ a.e., $A \in \mathcal{A}$ then

$$\alpha = E(f\bar{g}|\ \ ) \in L_\infty(X,\mathcal{A},m).$$

Proof

$$|E(f\bar{g}|\mathcal{A})| \leq E(|fg|^2|\mathcal{A})^{\frac{1}{2}} \quad \text{(Schwartz)}$$

$$\leq K\,\chi_A \text{ where } |f| \leq K$$

$$\leq K$$

Step 3

Now we are in a position to give the induction argument.

Let $g_1,\dots,g_{n-1}$ be functions such that $g_1,\dots,g_{n-1}$ are mutually $\mathcal{A}$-orthogonal, $\|g_i\|_{\mathcal{A}} = \chi_{A_i}$ a.e., $A_i \in \mathcal{A}$, $i = 1,\dots, n-1$ and $f_1,\dots,f_{n-1} \in \mathcal{A}$-span of $g_1,\dots,g_{n-1}$.

Consider $u_n = f_n - \alpha_1 g_1 \cdots\cdots \alpha_{n-1} g_{n-1}$ where $\alpha_i = E(f_n \bar{g}_i \,|\, \mathcal{A})$ a.e.. According to Step (2) the functions $\alpha_1,\dots,\alpha_{n-1}$ are bounded.

Define $g_n = \dfrac{u_n}{\|u_n\|_{\mathcal{A}}}$ on $A_n = \{x : \|u_n\|_{\mathcal{A}} \neq 0\}$

$= 0$ elsewhere.

Again we have $\|u_n\|_{\mathcal{A}}\, g_n = u_n$, and we note that $\|u_n\|_{\mathcal{A}}$ is bounded since

$$|f_n - \alpha_1 g_1 \cdots\cdots \alpha_{n-1} g_{n-1}| \leq 2^{\frac{1}{2}}\{\,|f_n|^2 + |\alpha_1 g_1 + \cdots + \alpha_{n-1} g_{n-1}|^2\}^{\frac{1}{2}}$$

and therefore

$$E(|f_n - \alpha_1 g_1 \cdots\cdots \alpha_{n-1} g_{n-1}|^2 \,|\, \mathcal{A})$$

$$\leq 2\, E\,(|f_n|^2 + |\,\alpha_1 g_1 + \cdots + \alpha_{n-1} g_{n-1}|^2 \,|\, \mathcal{A})$$

$$\leq 2\, E\,(K^2 + |\alpha_1|^2 \chi_{A_1} + \cdots + |\alpha_{n-1}|^2 \chi_{A_{n-1}})$$

which is bounded.

Hence $f_n = \|u_n\|_{\mathcal{A}}\, g_n + \alpha_1 g_1 + \cdots + \alpha_{n-1} g_{n-1}$

So that $f_1,\dots,f_n$ belong to the $\mathcal{A}$-span of $g_1,\dots,g_n$.

Clearly $E(f_n \bar{g}_i \,|\, \mathcal{A}) = \|u_n\|_{\mathcal{A}}\, E(g_n \bar{g}_i \,|\, \mathcal{A})$

$\pm\, \alpha_i\, \chi_{A_i}$

$= \alpha_i\, \chi_{A_i}$.

So that $E(g_n \bar{g}_i \,|\, \mathcal{A}) = 0$, for $i \neq n$.

Moreover

$$E(|g_n|^2|\mathcal{Q}) = E\left(\frac{|u_n|^2}{\|u_n\|_{\mathcal{Q}}} |\mathcal{Q}\right) = \chi_{A_n}.$$

This completes the proof.

Lemma 4.8.

Let $\{g_i\}_{i\in I}$ and $\{A_i\}_{i\in I}$ be constructed as above. Then, modified families $\{h_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ can be constructed with the following properties

i) $C_i \supset C_{i+1}$, $i \in I$

ii) $\langle h_i, h_j\rangle_{\mathcal{Q}} = 0$ a.e. $i \neq j$

$\|h_i\|_{\mathcal{Q}}^2 = \langle h_i, h_i\rangle_{\mathcal{Q}} = \chi_{C_i}$ a.e. $i = j$, $i,j \in I$

iii) The family $\{h_i\}_{i\in I}$ $\mathcal{Q}$-span $L^2(X,\mathcal{B},m)$

Proof

The proof will be sketched.

The sets $A_1, A_2, \ldots$ are the supports of the functions $E(|g_1|^2|\mathcal{Q})$, $E(|g_2|^2|\mathcal{Q})$, ... , so that the supports of g_1, $\chi_{A_2-A_1}\, g_2$, $\chi_{A_3-(A_1\cup A_2)}\, g_3$, ... are mutually disjoint.

Hence

$$h_1 = g_1 \pm \chi_{A_2-A_1}\, g_2 + \chi_{A_3-(A_1\cup A_2)}\, g_3 + \cdots\cdots$$

together with $\chi_{A_1\cap A_2}\, g_2$, $\chi_{A_3\cap A_1\cup A_2}\, g_3$, ...

have the same $\mathcal{Q}$-span as $g_1, g_2, \ldots\ldots$. Note that $\{x : \|h_1\|_{\mathcal{Q}} \neq 0\} = \cup A_n$

Hence we have arrived at an equivalent family of functions and sets $h_1, g_2^1, g_3^1, \ldots ; C_1, A_2^1, A_3^1, \ldots\ldots$ such that $C_1 \supset A_i^1$, $i = 2,3,\ldots$. We now repeat this procedure with $g_2^1, g_3^1, \ldots ; A_2^1, A_3^1, \ldots$ to obtain $h_2, g_3^2, \ldots ; C_2, A_3^2, \ldots$ where $C_2 \supset A_i^2$, $i = 3,4,\ldots$. This leads ultimately to the families

$$h_1, h_2, h_3, \ldots ; C_1, C_2, C_3, \ldots \text{ with } C_1 \supset C_2 \supset C_3 \supset \ldots\ldots$$

such that $g_1, g_2, \ldots$ and $h_1, h_2, \ldots$ have the same $\mathcal{A}$-span.

Lemma (4.7) and (4.8) will be combined to give the following theorem:

Theorem 4.9.

Given $\mathcal{A} \subset \mathcal{B}$ there exist a family of functions $\{h_i\}_{i\in I} \subset L^2(X,\mathcal{B},m)$ and a family $\{C_i\}_{i\in I}$ of sets in $\mathcal{A}$ such that

i) $\langle h_i h_j \rangle_{\mathcal{A}} = 0$ a.e. $i \neq j$, $i,j \in I$

$\|h_i\|_{\mathcal{A}} = \langle h_i, h_i \rangle_{\mathcal{A}} = \chi_{C_i}$ a.e., $C_i \supset C_{i+1}$, $i \in I$.

ii) the family $\{h_i\}_{i\in I}$ $\mathcal{A}$-span $L^2(X,\mathcal{B},m)$.

§5. A multiplicity pair for $M_{m_1} \otimes M_{m_2}$

It has been proven (cf §3), that the operator $M_{m_1} \otimes M_{m_2}$ in $L^2(m_1 \times m_2)$. $\emptyset$. Also, it has been shown that $m_1 * m_2$ is a maximal spectral type of $M_{m_1} \otimes M_{m_2}$, hence it is a maximal spectral type of W, as well. By applying theorem (4.9) to the space $L^2(m_1 \times m_2)\circ \emptyset)$, it can be decomposed, through a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B} \times \mathcal{B}$, into $\mathcal{A}$-orthogonal pieces (hence orthogonal pieces). Let $\{h_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ be the families so obtained for $L^2((m_1 \times m_2)\circ \emptyset)$.

It will turn out, that a particular choice of $\mathcal{A}$ will help in having the right decomposition so that a multiplicity function, say, $m_1 \circ m_2$, will be constructed for W and hence for $M_{m_1} \otimes M_{m_2}$. In fact, all what is needed, to construct such a multiplicity function, are the mutal $\mathcal{A}$-orthogonality of the h_i's, the properties of the C_i's and some known functional analysis facts.

Definition 5.1.

Consider the measurable space $(K \times K, \mathcal{B} \times \mathcal{B})$ where $K \times K$ is the two dimensional torus, $\mathcal{B}$ is the Borel-σ-algebra of K and $\mathcal{B} \times \mathcal{B}$ is the product σ-algebra. The family $\mathcal{A}$ of subsets of $K \times K$ of the form $K \times B$, where $B \in \mathcal{B}$, is a sub-σ-algebra of $\mathcal{B} \times \mathcal{B}$ which is isomorphic to the σ-algebra $\mathcal{B}$ of Borel sets in K (this fact will be used throughout without mentioning it). From now on $\mathcal{A}$ will be referred to as this σ-algebra. In fact this is the particular choice mentioned above.

Theorem 5.2.

Consider the unitary operators $M_{m_1} \otimes M_{m_2}$ in $L^2(K \times K, m_1 \times m_2)$, W in $L^2(K \times K, (m_1 \times m_2)\circ \emptyset)$ and $\mathcal{A}$ the above defined sub-σ-algebra of $\mathcal{B} \times \mathcal{B}$, then

I) The unitary operator W in $L^2((m_1 \times m_2)\circ\emptyset)$ is equivalent to the unitary operator $M_{m_1} \otimes M_{m_2}$ in $L^2(m_1 \times m_2)$.

II) $m_1 * m_2$ is a maximal spectral type of $M_{m_1} \otimes M_{m_2}$ and hence it is a maximum spectral type of W as well.

III) there exist families $\{h_i\}_{i\in I}$ and $\{C_i\}_{i\in I} \subset \mathcal{Q}$ such that

i) $\langle h_i, h_j\rangle_{\mathcal{Q}} = 0$ a.e. $i \neq j$, $i,j \in I$

$\|h_i\|^2_{\mathcal{Q}} = \langle h_i h_i\rangle_{\mathcal{Q}} = \chi_{C'_i}$ a.e. $i \in I$

ii) $C'_i = K \times C_i$, $C_i \supset C_{i+1}$, $i \in I$

iii) $\{h_i\}_{i\in I}$ $\mathcal{Q}$-span $L^2((m_1 \times m_2)\circ\emptyset)$

IV) $m_1 \circ m_2 = \Sigma\, \chi_{C_i}$ is a multiplicity function of W and hence it is a multiplicity function of $M_{m_1} \otimes M_{m_2}$. Consequently $(m_1 * m_2, m_1 \circ m_2)$ is a multiplicity pair for $M_{m_1} \otimes M_{m_2}$.

Proof

I) c.f. Lemma 2.2.

II) c.f. Lemma 3.2.

III) The families $\{h_i\}_{i\in I}$ and $\{C'_i\}_{i\in I}$ are the result of applying theorem (4.9) to the space $L^2((m_1 \times m_2)\circ\emptyset)$ with $\mathcal{Q}$ as the sub-σ-algebra.

IV) To construct a multiplicity function for a unitary operator in a Hilbert space one needs to decompose the space into mutually orthogonal cycles, say $Z(x_i)$ where x_i is a generator and such that

$\tilde{x}_{i+1} << \tilde{x}_i \quad i$ where $\tilde{x}_i$ is a spectral type x_i.

We have $h_1, h_2, \ldots\ \mathcal{Q}$-span $L^2 K\times K, \mathcal{B}\times\mathcal{B}, \mu)$ where $\mu = (m_1 \times m_2 \circ \emptyset)$

and $C_1 \supset C_2 \supset \ldots\ldots$ and

$$\langle h_i, h_j \rangle_{\mathcal{Q}} = E(h_i \bar{h}_j | \mathcal{Q}) = 0 \qquad i \neq j$$

$$= \chi_{C_i} \qquad i = j.$$

Let $Z(h_i)$ be the cycle generated by h_i under the operator W then

$Z(h_i) \perp Z(h_j)$ $i \neq j$. Since

$P(w)h_i \perp q(w)h_j$ $i \neq j$, $P(w)$, $q(w)$ are polynomials and

this follows from

$$\int P(w)\ \bar{q}(w)\ h_i \bar{h}_j\ d\mu$$

$$= \int P(w)\ \bar{q}(w)\ E\ (h_i \bar{h}_j | \mathcal{Q})\ d\mu = 0.$$

The cycle $Z(h_i) = \mathcal{Q}$-span of h_i since it is equal to $\{P(w)h_i$, P polynomial$\}$ and $\{P(w)$: P polynomial$\}$ is dense in $C(K)$.

Now the spectral types of h_i is given by

$$\langle W^n h_i, h_i \rangle = \iint_{K\times K} w^n\ |h_i|^2\ d\mu$$

$$= \iint_{K\times K} w^n\ E(|h_i|^2 | \mathcal{Q})\ d\mu$$

$$= \iint_{K\times K} w^n\ \chi_{C_i}\ d((m_1 \times m_2)\circ\emptyset)$$

$$= \iint_{K\times K} z^n\ w^n\ \chi_{C_i}\ (zw)\ d\ m_1 \times m_2$$

$$= \int_K w^n\ \chi_{C_i}\ d\ m_1 * m_2\ .$$

i.e. a spectral type $\tilde{h}_i$ of h_i is $m_1 * m_2 | C_i$.

Since $C_i \supset C_{i+1}$ $\forall$ i we have $\tilde{h}_{i+1} << \tilde{h}_i$ where $\tilde{h}_i$ is a spectral type of h_i.

It can be easily seen now that the restriction of the operator W to the cycle $Z(h_i)$ is canonically represented as the multiplication operator in $L^2(C_i, m_1 * m_2)$. Therefore $m_1 \circ m_2 = \Sigma \chi_{C_i}$ is a multiplicity function for W and hence for $M_{m_1} \otimes M_{m_2}$ by equivalence.

§6. A multiplicity pair for $U \otimes V$.

Let U and V be unitary operators in the Hilbert spaces H_1 and H_2 respectively. Let μ and ν be a maximal spectral type of U and V respectively. Then U and V have the following ordered representations

$$U \simeq \sum_i \oplus M_{\mu_i} \text{ in } H_1 \simeq \sum_i \oplus H_{\mu_i}, \text{ where}$$

$\mu_1 = \mu$, $\mu_i >> \mu_{i+1}$ for all i, $H_{\mu_i} = L^2(K,\mu_i)$ and M_{μ_i} is the multiplication operator in $L^2(K,\mu_i)$,

$$V \simeq \sum_j \oplus M_{\nu_j} \text{ in } H_2 \simeq \sum \oplus H_{\nu_j}, \text{ where } \nu_1 = \nu,\ \nu_j >> \nu_{j+1}$$

for all j, $H_{\nu_j} = L^2(K,\nu_j)$.

Consequently,

$$* \quad U \otimes V \simeq \bigoplus_{i,j} M_{\mu_i} \otimes M_{\nu_j} \text{ in } H_1 \otimes H_2 \simeq \bigoplus_{i,j} H_{\mu_i} \otimes H_{\nu_j}.$$

Obviously this is not an ordered representation i.e. one which leads to the construction of a multiplicity function. Using theorem 5.2, a multiplicity pair, for $M_{\mu_i} \otimes M_{\nu_j}$ can be defined. To derive a multiplicity for $U \otimes V$ out of the multiplicity pairs $(\mu_i * \nu_j, \mu_i \circ \nu_j)$, we shall need to introduce, what will be called calculus of multiplicity pairs.

<u>Calculus of multiplicity pairs</u>.

Let μ, ν be two Borel measures on K such that $\mu << \nu$, then there exist a Borel set E such that $\mu \sim \mu'$, where μ' is the measure defined as $\mu'(A) = \nu(E \cap A)$, for all $A \in \mathcal{B}$. The set E is unique up to sets of ν-measure zero.

Definition 6.1.

If μ and ν are two Borel measures and E is the set described above, the χ_E will be denoted by $\frac{\partial\mu}{\partial\nu}$

Definition 6.2.

Let MP be the set of all multiplicity pairs. Define the relation "~" in MP as follows:

$(\mu,f) \sim (\nu,g)$, $(\mu,f);(\nu,g) \in MP$ if $\exists$ another measure λ with $\lambda >> \mu$ and $\lambda >> \nu$ such that

$$\frac{\partial\mu}{\partial\lambda} f = \frac{\partial\nu}{\partial\lambda} g \quad \text{a.e.} \quad [\lambda] \qquad (1)$$

Remark

If (1) holds for a measure λ then it holds for any measure λ_1 that dominates λ. Note that the set E for μ is defined by the relation $\mu \sim \mu'$ where $\mu'(A) = \lambda\ (A \cap E)$ for all A.

Thus

$\mu'(A) = \int_A \chi_E \, d\lambda$ for all A. In other words

$\chi_E = \frac{d\mu'}{d\lambda}$ (the Radon-Nikodym derivative)

i.e. $\frac{\partial\mu}{\partial\lambda}$ is the radon Nikodym derivative of the measure μ' which is equivalent to μ, with respect to λ. Hence $\frac{\partial\mu}{\partial\lambda}$ follows the chain rule, or if $\lambda << \lambda_1$ then $\frac{\partial\mu}{\partial\lambda_1} = \frac{\partial\mu}{\partial\lambda} \frac{\partial\lambda}{\partial\lambda_1}$ a.e. $[\lambda_1]$.

Therefore, the claim is justified.

<u>Lemma</u>

"$\sim$" is an equivalence relation.

<u>Proof</u>

First "$\sim$" is symmetric and reflexive by definition. To prove transitivity, let $(\mu,f),(\nu,g)$ and $(\xi,u) \in MP$ such that $(\mu,f) \sim (\nu,g)$ and $(\nu,g) \sim (\xi,u)$. Hence $\exists$ a measure λ with $\mu << \lambda$ and $\nu << \lambda$ such that

$\frac{\partial\mu}{\partial\lambda} f = \frac{\partial\nu}{\partial\lambda} g$ a.e. $[\lambda]$ (1) and $\exists$ a measure λ_1 with $\nu << \lambda_1$ and $\xi << \lambda_1$ such that

$\frac{\partial\nu}{\partial\lambda_1} g = \frac{\partial\xi}{\partial\lambda_1} u$ a.e. $[\lambda_1]$ (2). By remark above (1) holds for any λ_2 with $\lambda << \lambda_2$ (say $\lambda_2 = \lambda + \lambda_1$) i.e.

$\frac{\partial\mu}{\partial\lambda_2} f = \frac{\partial\nu}{\partial\lambda_2} g$ a.e. $[\lambda_2]$(3). Also (2) holds for λ_2 since $\lambda_1 << \lambda_2$ i.e.

$$\frac{\partial\nu}{\partial\lambda_2} g = \frac{\partial\xi}{\partial\lambda_2} u \quad \text{a.e. } [\lambda_2](4).$$

From (3) and (4) we have

$$\frac{\partial\mu}{\partial\lambda_2} f = \frac{\partial\xi}{\partial\lambda_2} u \quad \text{a.e. } [\lambda_2]\ (5).$$

Therefore since λ_2 dominates both μ and ξ and (5) holds $(\mu,f) \sim (\xi,u)$. This completes the proof.

<u>Definition 6.4</u>.

Let (μ,f) and $(\nu,g) \in MP$. Define

(i) $(\mu,f) \oplus_{MP} (\nu,g) = (\lambda, f \frac{\partial\mu}{\partial\lambda} + g \frac{\partial\nu}{\partial\lambda})$ where λ is any measure with $\lambda >> \nu$ and $\lambda >> \mu$ and $\frac{\partial\mu}{\partial\lambda}$ and $\frac{\partial\nu}{\partial\lambda}$ as before (From now on multiplicity pairs which are equivalent will be regarded as equal).

By remark above this definition is justified.

(ii) $(\mu,1) \otimes_{MP} (\nu,1) = (\mu * \nu, \mu \,\dot{\circ}\, \nu)$

A unitary operator is characterized by a multiplicity pair i.e. equivalence of operator implies equivalence of multiplicity pairs. Moreover the map which takes a unitary operator to a multiplicity pair preserves the operations of direct sums and tensor products. Therefore the distributive law valid for unitary operators is valid for multiplicity pairs, as well.

<u>Theorem 6.5</u>.

Let U and V be unitary operators in the Hilbert spaces H_1 and H_2 respectively. Let (μ,f) and (ν,g) be the respective multiplicity pairs associated with U and V. Then, the unitary operator $U \otimes V$ in the Hilbert space $H_1 \otimes H_2$ has $(\mu * \nu, (\mu \,\dot{\circ}\, \nu)fg)$ as its multiplicity pair.

<u>Proof</u>.

Equivalence of operators implies equivalence of their multiplicity pairs. Thus, the equation *, above, becomes

$$\{ \underset{i}{\oplus}_{MP} (\mu_i, 1)\} \otimes_{MP} (\underset{j}{\oplus} \nu_j, 1)\}$$

$$= \underset{i,j}{\oplus} \{\mu_i, 1\} \otimes_{MP} \{\nu_j, 1\} = \underset{i,j}{\oplus} (\mu_i * \mu_j, \mu_i \mathbin{\dot{\circ}} \mu_j)$$

But $(\mu_i * \nu_j, \mu_i \mathbin{\dot{\circ}} \mu_j) = (\mu * \nu, (\mu \mathbin{\dot{\circ}} \nu) \frac{\partial\mu_i}{\partial\mu} \frac{\partial\nu_i}{\partial\nu})$.

Therefore,

$$\underset{i,j}{\oplus}_{MP} \{\mu_i * \nu_j, \mu_i \mathbin{\dot{\circ}} \nu_j\}$$

$$= \underset{i,j}{\oplus} \{\mu * \nu, (\mu \mathbin{\dot{\circ}} \nu) \frac{\partial\mu_i}{\partial\mu} \frac{\partial\nu_i}{\partial\nu} \}$$

$$= (\mu * \nu, (\mu \mathbin{\dot{\circ}} \nu) fg),$$

by definition of $\oplus_{MP}$ and since

$$f = \sum_i \frac{\partial\mu_i}{\partial\mu} \quad \text{and} \quad g = \sum_j \frac{\partial\nu_j}{\partial\nu} \cdot$$

7. Generalizations

The results of §6 have a natural generalization to the case of more than two unitary operators. Consider the operator $\bigotimes_{i=1}^{n} M_{m_i}$ in $L^2(K^n, \mathcal{B}^n, \underset{i=1}{\overset{n}{X}} m_i)$, where $K^n = K \times \ldots\ldots \times K$, n times, $\mathcal{B}^n$ is the product σ-algebra and $\underset{i=1}{\overset{n}{X}} m_i$ is the product measure. It has been proved in lemma 2.3 that a maximal spectral type of $\bigotimes_{i=1}^{n} M_{m_i}$ is $\mathop{*}_{i=1}^{n} m_i = m_1 * \ldots * m_n$. Also, it has been shown that the operator $\bigotimes_{i=1}^{n} M_{m_i}$ is equivalent to the operator W_n in $L^2(K^n, (\underset{i=1}{\overset{n}{X}} m_i)\circ\emptyset_n)$ through the isometry $A_{\emptyset_n}$ which is induced by the point map $\emptyset_n$, in lemma 3.4. In §5, the important tool was theorem 4.9. That will be so here, but with an appropriate sub σ-algebra $\mathcal{A}_n$ of $\mathcal{B}^n$. Now, we proceed to give the formal exposition.

Definition 7.1.

Consider the measurable space $(K^n, \mathcal{B}^n)$ where $K^n = K \times \ldots \times K$ = n-dimensional torus, $\mathcal{B}^n$ is the product Borel σ-algebra and $\mathcal{B}$ the Borel σ-algebra of Borel sets in $K = \{z \in \mathbb{C} \mid |z| = 1\}$. The family $\mathcal{A}_n$ of subsets of K^n of the form $\underbrace{K \times \ldots\ldots \times K}_{n-1} \times \mathcal{B}$ is a sub-σ-algebra $\mathcal{A}_n$ of $\mathcal{B}_n$ which is isomorphic to the σ-algebra $\mathcal{B}$ of Borel sets in K. From now on $\mathcal{A}_n$ will mean this σ-algebra.

Theorem 7.2.

Consider the unitary operator $\overset{n}{\underset{k=1}{\otimes}} M_{m_k}$ in $L^2(K^n, \overset{n}{\underset{k=1}{X}} m_k)$, the unitary operator W_n in $L^2(K^n, (\overset{n}{\underset{k=1}{X}} m_i) \circ \emptyset_n)$ and the sub-σ-algebra $\mathcal{A}_n$ of $\mathcal{B}^n$. Then

I) The unitary operator W_n in $L^2(K^n, (\overset{n}{\underset{k=1}{X}} m_k) \circ \emptyset_n)$ is equivalent to the unitary operator $\overset{n}{\underset{i=1}{\otimes}} M_{m_i}$ in $L^2(K^n, \overset{n}{\underset{k=1}{X}} m_k)$

II) $m_1 * \ldots * m_n = \overset{n}{\underset{k=1}{*}} m_k$ is a maximal spectral type of $\overset{n}{\underset{k=1}{\otimes}} M_{m_k}$

and hence it is a maximal spectral type of W_n.

III) There exist families $\{h_i\}_{i\in I} \subset L_\infty(\overset{n}{\underset{k=1}{X}} m_i \circ \emptyset)$

and $\{C_i\}_{i\in I} \subset \mathcal{A}_n$ such that

(i) $\langle h_i, h_j \rangle_{\mathcal{A}_n} = 0$ a.e. $i \neq j$

$\|h_i\|_{\mathcal{A}_n} = \langle h_i, h_i \rangle_{\mathcal{A}_n} = \chi_{C_i}$ a.e., $i\in j, i, j \in g$

(ii) $C_i \supset C_{i+1}$ $i \in I$

(iii) $\{h_i\}_{i\in I}$ $\mathcal{A}_n$-span $L^2((\overset{n}{\underset{k=1}{X}} m_k) \circ \emptyset_n)$

IV) $\overset{n}{\underset{k=1}{\dot{O}}} m_k = \sum_i \chi_{C_i}$ is a multiplicity function for W_n and hence it is a multiplicity pair for $\overset{n}{\underset{k=1}{\otimes}} M_{m_k}$. Consequently $(\overset{n}{\underset{k=1}{*}} m_k, \overset{n}{\underset{k=1}{\dot{O}}} m_k)$ is a multiplicity pair for $\overset{n}{\underset{k=1}{\otimes}} M_{m_k}$.

Proof

I) c.f. lemma 3.4.

II) c.f. lemma 2.3.

III) The families $\{h_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ can be obtained by applying theorem 4.9. to the space $L^2((\overset{n}{\underset{k=1}{X}} m_k)\circ \phi_n)$ with $\mathcal{Q}_n$ as the sub-σ-algebra.

IV) The proof is the same as the proof of theorem 5.2. part IV, with the appropriate changes.

Now, a generalized version of theorem 6.5 can be given but before that, the following is needed.

Let U_ℓ, $\ell=1,\ldots,s$ be unitary operators in the Hilbert space H_ℓ, $\ell=1,\ldots,s$ respectively and let μ_ℓ, $\ell=1,\ldots,s$ be maximal spectral types of U , $= 1,\ldots,s$ respectively. Then for each ℓ , $\ell = 1,\ldots,s$, U_ℓ has the following ordered representation

$$U_\ell = \sum_i \oplus M_{\mu_\ell^i} \text{ in } H_\ell \simeq \sum_i \oplus H_{\mu_\ell^i}$$

where $\mu_\ell^1 = \mu_\ell$, $\mu_\ell^i >> \mu_\ell^{i+1}$ for all i and $H_{\mu_\ell^i} = L^2(K, \mu_\ell^i)$ and $M_{\mu_\ell^i}$ is the multiplication operator in $L^2(K, \mu_\ell^i)$.

Consequently

$$* \overset{s}{\underset{\ell=1}{\otimes}} U \simeq \underset{i_1,\ldots,i_s}{\oplus} \overset{s}{\underset{\ell=1}{\otimes}} M_{\mu_\ell^i} \quad \text{in}$$

$$\overset{s}{\underset{\ell=1}{\otimes}} H_\ell \simeq \underset{i_1,\ldots,i_s}{\oplus} \overset{s}{\underset{\ell=1}{\otimes}} H_{\mu_\ell^i} .$$

Theorem 7.3.

Let U_ℓ, $\ell = 1,\ldots,s$ be unitary operators in the Hilbert spaces H_ℓ, $\ell = 1,\ldots,s$ respectively. Let (μ_ℓ, f_ℓ) be a respective multiplicity pair for U_ℓ, $\ell = 1,\ldots, s$ respectively. Then the unitary operator $\bigotimes_{\ell=1}^{s} U_\ell$ in the Hilbert space $\bigotimes_{\ell=1}^{s} H$ has

$$\left(\mathop{*}_{\ell=1}^{s} \mu \ , \ \left(\mathop{\circ}_{\ell=1}^{s} \mu\right)(f_1 \times \ldots \times f_s)\right)$$

as a multiplicity pair.

Proof.

The proof is a generalized version of the proof of theorem 6.5.

§8. Categories

The present section is a rather short one, although, big structures will be employed throughout. It can be looked at as a summing up of what has been, hitherto, accounted for. The big structures are categories, more precisely, two categories. The category of measure spaces and the category of Hilbert spaces.

Definition 8.1.

The category $\mathfrak{X}$ of measure spaces is defined as follows

Objects: Measure spaces

Morphisms: Homomorphisms

Composition Law: Ordinary composition of maps

Definition 8.2.

The category $\mathcal{H}$ of Hilbert spaces is defined as follows

Objects: Hilbert spaces

Morphisms: Linear operators

Composition Law: Ordinary composition of maps.

There is a natural functorial relation between the categories $\mathcal{X}$ and $\mathcal{H}$ through a functor F which takes each measure space X to its L^2-space $L^2(X)$ and the homomorphism to its induced linear operator. If X is a Lebesgue space and the homomorphism is an invertible measure preserving transformation then the induced linear operator is unitary (isometry). If W is the group of invertible measure preserving transformation on a Lebesgue space, then one of the conjugacy invariants is the spectral invariant. However this is not a complete invariant. The relation between conjugacy invariants and spectral invariants is in fact the functor F.

The functor F takes the operations disjoint unions in $\mathcal{X}$ to orthogonal direct sums in $\mathcal{H}$ and direct products in $\mathcal{X}$ to tensor products in $\mathcal{H}$. Moreover, a unitary operator can be looked at as a multiplicity pair and again F preserves these operations. We shall not pursue that line of thought any further since it does not seem to throw more light on the structure of $\mathcal{X}$.

§9. Tensor products of operators with discrete spectrum.

Let U be a unitary operator in a Hilbert space H with simple discrete spectrum. Then U is equivalent to the multiplication operator M_μ in $L^2(K,\mu)$, where μ is maximal spectral type of U. Also the unitary operator $M_\mu \otimes M_\mu$ in $L^2(K \times K, \mu \times \mu)$ is equivalent to the operator $U \otimes U$ in $H \otimes H$. A multiplicity pair could be assigned to the unitary operator $U \otimes U$, through theorem 5.2, namely, $(\mu * \mu, \mu \dot{o} \mu)$. The symmetric product $U \odot U$ is the restriction of $U \otimes U$ to the symmetric subspace $H \odot H$ of $H \otimes H$. $U \odot U$ has $\mu * \mu$ as a maximal spectral type, see later. Let $U \odot U$ as a multiplicity pair $(\mu * \mu, f)$. In this section a formula will be given through which $(\mu * \mu, f)$ could be calculated.

Notation

I) H_s^o = subspace spanned by symmetric functions in $L^2(K \times K)$ with support off the diagonal. (A function $f \in L^2(K \times K)$ is symmetric if $f(z,w) = f(w,z)$ and the diagonal is the set $(z,z) \in K \times K$. The restriction of $M_\mu \otimes M_\mu$ to H_s^o will be denoted by V_s^o.

II) H_d = subspace of $L^2(K \times K)$ with support on the diagonal. The restriction of $M_\mu \otimes M_\mu$ to H_d will be denoted by V_d.

III) H_a = subspace of $L^2(K \times K)$ spanned by antisymmetric functions in $L^2(K \times K)$. (A function $f \in L^2(K \times K)$ is antisymmetric if $f(z,w) = -f(w,z)$.

It is clear that H_s^o is orthogonal to H_d. The orthogonal direct sum of H_s^o and H_d will be denoted H_s. In fact V_s, the restriction of $M_\mu \otimes M_\mu$ to H_s is equivalent to $U \odot U$ in $H \odot H$ (about the definition of $U \odot U$ and $H \odot H$ see Ch.3).

Also H_a is orthogonal to H_s since $(f,g)=-(f,g)$, $f \in H_a$, $g \in H_s$. Moreover $L^2(K \times K)$ is the direct sum of H_a and H_s, since, if $f \in L^2(K,K)$ then $f = f_1 + f_2$ where $f_1(z,w) = \dfrac{f(z,w) + f(w,z)}{2}$ and $f_2 = \dfrac{f(z,w)-f(w,z)}{2}$.

$K \times K$ could be looked at as the unit square mod 1 in R^2. The diagonal d of $K \times K$ corresponds to the diagonal of the unit square. Let Δ be subset of $K \times K$ which corresponds to the area above the diagonal in the unit square while Δ' denotes the subset of $K \times K$ which corresponds to the area below. Then if $f \in H_a$ let $g = \chi_\Delta f - \chi_{\Delta'} f$. The claim now that $g \in H_s$. For $g(z,w) = g(w,z)$, since $\chi_\Delta(z,w)=\chi_{\Delta'}(w,z)$. This correspondence is an equivalence between H_s and H_a which takes V_s to V_a.

Let λ be an eigenvalue of U with eigen function f. Then $U^2 f = \lambda^2 f$ i.e. U^2 is a unitary operatory with simple discrete spectrum $\{\lambda^2\}_{\lambda\in\Lambda}$ where Λ is the spectrum of U. In fact V_d is isomorphic to U^2 since H_d is spanned by f^2, f is an eigenvalue of U. In other words the maximal spectral type of V_d is $\mu * \mu$, its multiplicity is one.

Finally, from the following equation

$$(\mu * \mu,\ \mu \circ \mu) = (\mu * \mu, f) \ominus_{MP} (\mu * \mu; f) \ominus_{MP} (\mu * \mu \quad , 1)$$

f could be computed.

CHAPTER III

GAUSSIAN PROCESSES

A unitary operator in a gaussian subspace has a multiplicative extension to the bigger function space. The latter is equivalent to the symmetric Fock space. The equivalence will help in the spectral analysis of the multiplicative extensions. The object of this Chapter is to develop the theory beginning with a gauss process and ending with a multiplicity pair. ([7],[11] and [20])

§0. General facts about stochastic processes. Let $(\Omega, \mathcal{B}, P)$ be a prob. space. A family of random variables $(X_t, t\in T,\ T \subset R)$ is an indexing set, is called a stochastic process. By a random variable, it is meant, a real valued measurable function on $(\Omega, \mathcal{B}, P)$ or more precisely $X_t: (\Omega, \mathcal{B}, P) \to (R, \mathcal{B}(R))$ where $\mathcal{B}(R)$ is the Borel σ-field on the real line. However, the values of a random variable can be taken in any measurable space. Nevertheless, most of the work here deals with real valued random variables.

A stochastic process which is indexed by the set of integers is called a discrete time process.

The expectation $E(X)$ of a random variable X is $\int_X X dP$. $E(X^2)$ is called the variance of X and E (XY) the covariance of the r.v's X and Y, when $E(X) = E(Y) = 0$. The distribution P_X of a.r.v. X is a prob.measure defined on $(R, \mathcal{B}(R))$ by

$$P_X(B) = P(X^{-1}(B)) = (\omega \in \Omega : X(\omega) \in B),\ \forall\ B \in \mathcal{B}(R)$$

By generalizing this idea to more than one random, say $X_{t_1}, \ldots\ldots, X_{t_n}$ we get what is called the finite dimensional distribution. It is a prob. measure on $(R^n, \mathcal{B}(R^n))$ defined by

$$P_{t_1 \ldots\ldots t_n}(B) = P\left(\omega \in \Omega \mid (X_{t_1}(\omega), \ldots, X_{t_n}(\omega), \ldots\ldots, X_{t_n}(\omega)) \in B\right),$$

$$\forall\, B \in \mathcal{B}(R^n) = \text{Borel } \sigma\text{-field on } R^n.$$

A stochastic process $\{X_t,\ t \in T\}$ is stationary if $P_{t_1 \ldots\ldots t_n}$ is the same as $P_{t_1+h, \ldots\ldots, t_n+h}$, $h \in R$, i.e. the finite dimensional distribution are invariant under time translations.

Consider discrete time stochastic processes. For any stationary stochastic process on a Lebesgue space $(\Omega, \ ,P)$s.t.r.v's generate $\mathcal{B}$ $\exists$ a unique m.p.t.s.t. $X_n(\omega) = X_0(T^n\omega)$. It is known that m.p.t. induce operator U_T in $L^2(\Omega, \mathcal{B}, P)$ which is given by the relation

$$U_T f = f \circ T\ ,\ f \in L^2(\Omega, \mathcal{B}, P). \quad \text{Evidently } U_T X_n = X_{n+1}.$$

§1. Gaussian process and Gaussian subspace

Definition 1.1.

Gaussian processes.

A stochastic process $\{X_t,\ t \in T\}$ on $(\Omega, \mathcal{B}, P)$ is Gaussian if the finite dimensional distributions of the process are gaussian. A finite dimensional distribution of $X_{t_1}, \ldots\ldots, X_{t_n}$ is gaussian if

$$P_{t_1 \ldots\ldots t_n}(B) = \frac{|A|^{\frac{1}{2}}}{(2\Pi)^{n/2}} \int_B e^{-\frac{1}{2}\langle Ax,x\rangle}\, dx,$$

where

$$x = (x_1, \ldots\ldots, x_n) \in R^n,\ dx = dx_1 \ldots\ldots dx_n,$$

$A = \{a_{ij}\}$ is a +ve definite symmetric matrix,

$$\langle Ax,x\rangle = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j \text{ and } B \in \mathcal{B}(R^n)$$

<u>Definition 1.2.</u>

A closed subspace of $L^2(\Omega)$ is called Gaussian if each function in this subspace has a Gaussian distribution.

Let $\{X_n, n \geq 0\}$ be a discrete time Gaussian Process. The closure H of the linear span of r.v's defining the process is a Gaussian space. H is said to be fundamental if $\mathcal{B}$ is the smallest σ-algebra for which all the r.v.'s $\{X_n\}$ are measurable. In this case $\mathcal{B}(H)$ replaces $\mathcal{B}$.

<u>Definition 1.3.</u>

<u>Weak stationary.</u>

A discrete time Gaussian process $\{X_n, n \geq 0\}$ is weak stationary if $E(X_i\ X_j) = E(X_{i+1}\ X_{j+1})$ In this case weak stationary $\Rightarrow$ stationary.

<u>Definition 1.4.</u>

Towards the establishing of the final result some more structures generated by H, a Gauss subspace, are needed. First let $P_n(H)$ be the set of all polynomials in members of H with degree $\leq$ n. Let $P(H) = \bigcup_{n=0}^{\infty} P_n$, which can be seen as a ring and is called the ring generated by H.

Also $\exp H$ is the set of all maps $X \mapsto \exp X$ where $X \in H$ and $\exp X \in L^2(\Omega, \mathcal{B}(H), P)$. $R(H)$, $\exp H$ and $L^2(\Omega, \mathcal{B}(H), P)$ are related to each other by the following relations which will not be proved here:

$\exp H$ generates $L^2(\mathcal{B}(H))$, $\exp H \subset R(H)$ and $R(H)$ is dense in $L^2(\mathcal{B}(H))$.

§ 2. Symmetric Tensor Products.

The following is a summary of certain facts about symmetric tensor products of Hilbert spaces.

$H^{n\otimes}$ is known from Ch. II.

$H^{n\odot}$ = all symmetric element of H^n. It is a subspace of $H^{n\otimes}$. Let S_n be the symmetric gp and σ any permutation in S_n. Let $u_\sigma, \sigma \in S_n$ be the operator in $H^{n\otimes}$ defined by

$$v_1 \otimes \ldots\ldots \otimes v_n \longmapsto v_{\sigma(1)} \otimes \ldots\ldots \otimes v_{\sigma(n)}.$$

The operator u_σ is easily seen to be unitary. Moreover $\frac{1}{n!} \sum_{\sigma \in S_n} u_\sigma$ is an orthogonal projection of $H^{n\otimes}$ on H^n.

Definition 2.1.

1) $v_1 \circ \ldots\ldots \circ v_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} u_\sigma (v_1 \otimes \ldots\ldots \otimes v_n)$

2) $(v_1 \odot \ldots\ldots \odot v_n, w_1 \odot \ldots\ldots \odot w_n) = \sum_{\sigma \in S_n} (v_1, w_{\sigma(1)}) \ldots (v_n, w_{\sigma(n)})$

3) Since $v^{n\odot} = \sqrt{n!}\ v^{n\otimes}$ then $(v^{n\odot}, w^{n\odot}) = n!\ (v,w)^n$

Definition 2.3.

Symmetric tensor product of unitary operators.

Let U be a unitary operator in $H.U^{n\otimes}$ in $H^{n\otimes}$ has been defined before. The restriction of $U^{n\otimes}$ to $H^{n\odot}$ will be called written $U^{n\odot}$ and is defined as the restriction of $U^{n\otimes}$ to $H^{n\odot}$. $U^{n\odot}$ is unitary.

Definition 2.3.

Symmetric Fock space over H.

The countable direct sum $\sum_{i=0}^{\infty} H^{n\odot}$ where $H^{\circ\odot} = \mathbb{C}$ is called the symmetric Fock space over H and is denoted by $\operatorname{Exp} \odot H$ For every $v \in H$, write

$$\operatorname{Exp} v = 1 \oplus = \oplus \frac{1}{\sqrt{2!}} v^{2\odot} \oplus \ldots\ldots = \sum_{i=0}^{\infty} \frac{1}{\sqrt{n!}} v^{n\odot} .$$

$(\operatorname{Exp} v, \operatorname{Exp} v') = \operatorname{Exp} (v,v'),\ v,v' \in H$

It should be remarked that $\{\operatorname{Exp} v,\ v \in H\}$ span $\operatorname{Exp} \odot H$

Lemma 2.4.

Let $(v_i)_{i\in I}$ be an orthonormal basis for a Hilbert space H.

Let $(n_i)_{i\in I}$ be a summable family of integers.

Put

$v_{\{n_i, i\in I\}} = (\prod_{i\in I} n_i!)^{-\frac{1}{2}} \bigodot_I v_i^{n_i \odot}$ Then for any integer $n \geq 1$, the family $\{ v_{\{n_i, i\in I\}},\ \sum_I n_i = n\}$ is a basis for $H^{n\odot}$.

<u>Proof</u>.

See [16]

For any proof concerning the basis of H^{nc} it is enough to prove for $\left(\prod_{j=1}^{k} n_{i_k}\right)^{-\frac{1}{2}} \bigotimes_{j=1}^{k} v_{i_j}^{\otimes n_{i_j}}$

<u>Definition 2.5</u>.

<u>Hermite polynomials</u>.

Hermite polynomials $h_n(x)$ of degree n on R is defined from

$$\exp\left(u\,x - \frac{u^2}{2}\right) = \sum_{n\geq 0} \frac{u^n}{n!}\, h_n(x)$$

Also they are the polynomials arrived at by applying G-S process to $(1, x, x^2, \ldots)$ in $L^2\left(R, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx\right)$ where they form an orthonormal basis for it.

Also, we note here the following: $\exp\left(\sum_{i=1}^{k} u_i X_i - \frac{1}{2} \sum_{i=1}^{k} u_i^2 \sigma_i^{\,2}\right)$

$$= \prod_{1}^{k} \exp\left(u_i x_i - \tfrac{1}{2} u_i^2 \sigma_i^2\right) = \prod_{1}^{k} \sum_{n\geq 0} \frac{u_i^n \sigma_2^{\,n}}{n!}\; h_n\left(\frac{x_i}{\sigma_i}\right)$$

$$= \sum_{n\geq 0} \frac{u_1^{\,n_1} \cdots\cdots u_k^{\,n_k}}{n_1! \cdots\cdots n_k!}\; \sigma_1^{\,n_1} \cdots\cdots \sigma_k^{\,n_k}\; h_{n_1}\left(\frac{x_1}{\sigma_1}\right) \cdots\cdots h_{n_k}\left(\frac{x_k}{\sigma_k}\right) \qquad (*)$$

§3. The representation of $L^2(\Omega, \mathcal{B}(H), P)$

The representation of $L^2(\Omega, \mathcal{B}(H), P)$ as a symmetric Fock space will be given here:

Proof.

See [16]

For any proof concerning the basis of $H^{n\odot}$ it is enough to prove for $\left(\prod_{j=1}^{k} n_{i_k}\right)^{-\frac{1}{2}} \bigotimes_{j=1}^{k} v_{i_j}^{\odot n_{1_j}}$

Definition 2.5.

Hermite polynomials.

Hermite polynomials $h_n(x)$ of degree n on R is defined from

$$\exp(u\, x - \frac{u^2}{2}) = \sum_{n\geq 0} \frac{u^n}{n!} h_n(x)$$

Also they are the polynomials arrived at by applying G-S process to $(1,x,x^2,\ldots.)$ in $L^2\,(R,\frac{1}{\sqrt{2\Pi}}\;e^{-\frac{1}{2}x^2}\,dx)$ where they form an orthonormal basis for it.

Also, we note here the following: $\exp\left(\sum_{i=1}^{k} u_i X_i - \frac{1}{2}\sum_{i=1}^{k} u_i^2\sigma_i^2\right)$

$$= \prod_{1}^{k} \exp\,(u_i x_i - \tfrac{1}{2}\, u_i^2\, \sigma_i^2 = \prod_{1}^{k} \sum_{n\geq 0} \frac{u_i^n \sigma_2{}^n}{n!} \quad h_n\left(\frac{x_i}{\sigma_i}\right)$$

$$= \sum_{n\geq 0} \frac{u_1^{n_1} \ldots\ldots u_k^{n_k}}{n_1! \;\ldots\ldots\; n_k!} \;\sigma_1^{n_1} \ldots\ldots \sigma_k^{n_k}\; h_{n_1}\left(\frac{x_1}{\sigma_1}\right) \ldots\ldots h_{n_k}\left(\frac{x_k}{\sigma_k}\right) \qquad (*)$$

§3. The representation of $L^2\;(\Omega, \mathcal{B}(H), P)$

The representation of $L^2\;(\Omega, \mathcal{B}(H), P)$ as a symmetric Fock space will be given here:

Theorem 3.1.

If H is a Gauss subspace of L^2 $(\Omega, \mathcal{B}(H), P)$ then $\exists$ a unique isometry $\emptyset$ of $\exp \circ H$ on to L^2 $(\Omega, \mathcal{B}(H), P)$ s.t.

$$\emptyset\,(\exp \circ X) = \exp\,(X - \tfrac{1}{2}\,E^2(X)),$$

Recall that $E^2(X) = \sigma^2$, the variance of X.

Theorem 3.2.

Let $(X_i)_{i\in I}$ be an orthonormal in H then the family

$\{h_{\{n_i,\ \sum_I n_i = n\}}\}$, $n \geq 1$ is an orthonormal basis for L^2 $(\Omega, \mathcal{B}(H), P)$,

where

$$h_{\{n_i,\ \sum_I n_i = n\}} = (\prod_I n_k!)^{-\frac{1}{2}} \prod_I h_{n_k}(X_{i_k})$$

Here $h_o \equiv 1$.

Proof.

$$\operatorname{Exp} \odot\, (u_1 X_{i_1} + \ldots\ldots + u_k X_{i_k})$$

$$= \sum \frac{1}{n!}\,(u_1 X_{i_1} + \ldots\ldots + u_k X_{i_k})^{n\odot}$$

but

$$(u_1 X_{i_1} + \ldots\ldots + u_k X_{i_k})^{n\odot} = \sum_{n_{i_1}+\ldots+n_{i_k}=n} u_1^{n_{i_1}} \ldots\ldots u_k^{n_{i_k}} \binom{n}{n_{i_1} \ldots\ldots n_{i_k}} (X_{i_1}^{n_{i_1}\odot} \odot \ldots \odot X_{i_k}^{n_{i_k}\odot})$$

$$= \sum_{n\geq 0} \sum_{n_{i_1}+\ldots+n_{i_k}=n} \frac{u_1^{n_{i_1}} \ldots\ldots u_k^{n_{i_k}}}{n_{i_1}! \ldots n_{i_k}!} (X_{i_1}^{n_{i_1}\odot} \odot \ldots\ldots \odot X_{i_k}^{n_{i_k}\odot})$$

* in 2.5 and Theorem 3.1

$$\sigma_{i_1}^{n_{i_1}} h_{m_{i_1}}\left(\frac{X_{i_1}}{\sigma_{i_1}}\right) \cdots\cdots \sigma_{i_k}^{n_{i_k}} h_{n_{i_k}}\left(\frac{X_{i_k}}{\sigma_{i_k}}\right) \text{ is the } \emptyset \text{ image of}$$

$$X_{i_1}^{n_{i_1}\odot} \odot \cdots\cdots \odot X_{i_k}^{n_{i_k}\odot} .\qquad \text{Q.E.D.}$$

§4. Spectral Analysis.

Let T be the unique unitary operator on $(\Omega,\ \ ,P)$ s.t. $X_{n+1}(w) = X_n(Tw)$, $X_n \in H$, a Gauss subspace. Then the induced unitary operator U_T in $L^2\ (\Omega,\mathcal{B}(H),P)$ is the unique multiplicative (i.e. $U_T\ (fg) = U_T\ f\ U_T\ g$, f, $g \in L^2$) extension of $U_T|H$ where $U_T|H$ is the operator sending $X_n \to X_{n+1}$. $U_T|H$ is cyclic by definition.

Now we are in a position to state and prove the following.

Theorem 4.1.

Let U be a cyclic unitary operator in a Gauss subspace H. Then the multiplicative extension M of U to $L^2\ (\Omega,\mathcal{B}(H),P) = \overline{R(H)}$ is unitarily equivalent to $\sum_{n=0}^{\infty} U^{n\odot}$ on exp $\odot$ HU

Proof.

$$\begin{array}{ccc}
X_{i_1}^{n_{i_1}\odot} \odot \cdots\cdots \odot X_{i_k}^{n_{i_k}\odot} & \xrightarrow{\emptyset} & h_{n_{i_1}}(X_{i_1}) \cdots h_{n_{i_k}}(X_{i_k}) \\
U^{n\odot}\downarrow & & \downarrow M \\
(UX_{i_1})^{n_{i_1}\odot} \odot \cdots\cdots \odot (UX_{i_k})^{n_{i_k}\odot} & \xrightarrow{\emptyset} & h_{n_{i_1}}(UX_{i_1}) \cdots h_{n_{i_k}}(UX_{i_k})
\end{array}$$

The commutativity of the above diagram together with extension to all linear combination prove the theorem.

What is the maximum spectral type of U_T in $L^2(\Omega, \mathcal{B}(H), P)$? To answer this question we need the maximal spectral type of $U^{n\otimes}$ in $H^{n\otimes}$.

Let U be a cyclic unitary operator in H. Let $x \in H$ be a generator and therefore the type μ of x is maximal. Then $\mu^n (\mu^n = \underbrace{\mu * \ldots\ldots * \mu}_{k})$ is the maximal spectral type of $U^{n\otimes}$ in $H^{n\otimes}$.

But $\underbrace{x \otimes \ldots\ldots \otimes x}_{n} \in H$ then μ^n is the maximal spectral type of $U^{n\otimes}$ in $H^{n\otimes}$.

Hence, the answer to the given question is the following Theorem.

Theorem 4.2.

U_T in $L^2(\Omega, \mathcal{B}(H), P)$ has maximal spectral type $\sum_{n=0}^{\infty} \mu^n$, where $\mu^0 = \delta$ (the measure having the unit mass at the origin in K i.e. 1)

SOME INVARIANT σ-ALGEBRAS FOR MEASURE PRESERVING TRANSFORMATION.

§0 Introduction.

0.1. Let $(X, \mathcal{B}, m)$ be a Lebesgue space [24] and T a measure preserving transformation on $(X, \mathcal{B}, m)$. Then $(X, \mathcal{B}, m, T)$ will be called a dynamical system. Throughout the present chapter, a dynamical property of $(X, \mathcal{B}, m, T)$ will be attributed to T rather than the underlying system. U_T will denote the induced unitary operator in $L^2(X, \mathcal{B}, m)$. A spectral property of T means a spectral property of U_T.

T is said to be ergodic if $T^{-1} B = B$, $B \in \mathcal{B} \Rightarrow m(B) = 0$ or 1. T is weak mixing if $\forall A, B \in \mathcal{B}\ \frac{1}{n} \sum_{i=1}^{n} |m(T^{-i} A \cap B) - m(A)\, m(B)| \to 0 \Leftrightarrow$ $\forall f \in L^2(X, \mathcal{B}, m)$, $\frac{1}{n} \sum_{i=1}^{n} | \langle U_T^2 f, f\rangle - \langle f, 1\rangle \langle 1, f\rangle | \to 0$

$$\Leftrightarrow \frac{1}{n} \sum_{i=1}^{n} | \langle U_T^2 f, g\rangle - \langle f, 1\rangle \langle 1, g\rangle | \to 0 \ \forall f, g \in L^2(X, \mathcal{B}, m).$$

T is strong mixing if $\forall A, B \in \mathcal{B}$, $m(T^{-n} A \cap B) \to m(A)\, m(B) \Leftrightarrow \langle U_T^n f, f\rangle \to \langle f, 1\rangle \langle 1, f\rangle\ \forall f \in L^2(X, \mathcal{B}, m) \Leftrightarrow \langle U_T^n f, g\rangle \to \langle f, 1\rangle \langle 1, g\rangle\ \forall f, g \in L^2(X, \mathcal{B}, m)$. It is a well known result that T is strong mixing $\Rightarrow$ T is weak mixing $\Rightarrow$ T is ergodic and the converse is false. For more details see [10], [29].

0.2. In his paper [28], P. Walters defined, for an invertible measure preserving transformation of a Lebesgue space $(X, \mathcal{B}, m)$ and a sequence $N = \{n_i\}^{\infty}$, of integers, an invariant sub-σ-algebra $\mathcal{Q}_N \subset \mathcal{B}$ by

$\mathcal{A}_N(T) = \{A \in \mathcal{B} \mid m(T^{n_i} A \Delta A) \to 0\}$. (From now on, we shall adopt the notations $\theta = \{\theta(i)\}$ for a sequence of integers and $\mathcal{A}'_\theta$ for Walters σ-algebra). In the present work we shall take the idea further and define an invariant sub-σ-algebra by

$$\mathcal{A}_\theta(T) = \{ A \in \mathcal{B} \mid m(T^{\theta(i)} A \Delta T^{\theta(j)} A) \to \text{ as } i,j \to \infty\}.$$

It will be proved that $\mathcal{A}'_\theta(T) \subset \mathcal{A}_\theta(T)$ and that $\mathcal{A}'_\theta$ and $\mathcal{A}_\theta$ has properties in common.

Let W denotes the class of all invertible measure preserving transformation of a Lebesgue space $(X, \mathcal{B}, m)$; W is a topological group. The topology is the weak topology which is inherited from the weak topology in the group of unitary operators in $L^2(X,\mathcal{B},m)$. So $T_j \to T$ weakly in W iff $\langle U_{T_j} f,g\rangle \to \langle U_T f,g\rangle$ for each $f,g \in L^2(X)$. It is worth noticing that the strong topology, where $U_j \to U$ strongly in U iff $\| U_j f - Uf\| \to 0$ for all $f \in L^2(X,\mathcal{B},m)$, coincides with the weak topology. W is also a complete metric group (see [10]).

It has been stated in [28], that $\mathcal{A}'_\theta(T) = \mathcal{B}$ implies $T^{\theta(i)} \to I$ weakly (strongly) in W, where I is the identity transformation. It will be proved here that $\mathcal{A}_\theta(T) = \mathcal{B}$ implies that $T^{\theta(i)} \to S$ weakly (strongly) in W, $S \in W$. This will lead to the definition of a set $G(T) \subset W$ for every $T \in W$. $G(T)$ will be shown to be a commutative subgroup of W where the restriction of the metric in W to $G(T)$ is invariant. Furthermore, the subgroup $G(T)$ will be called the T-group.

The behaviour of the σ-algebras $\mathcal{Q}_\theta(T)$ under group extensions will be investigated. If T_μ is a Gaussian shift based on the covariance measure μ i.e. a shift generated by a stationary Gaussian process with covariance measure μ, then $G(T_\mu)$ will be studied. It will be shown that members of $G(T_\mu)$ are generalized Gaussian shifts b~~ased on covariance measur~~es relat~~ed to μ and~~ the limit points of (λ^n), $n \in N$, $\lambda \in K = \{z \in \mathbb{C} \mid |z|=1\}$, in $L^2(K,\mu)$. Moreover, if μ is concentrated on a kronecker set $\mathcal{X}$ of $I = [-\frac{1}{2},\frac{1}{2}] \subset \mathbb{R}$ then $G(T_\mu)$ will be isomorphic to $C(\mathcal{X},K)$ = continuous functions with support on K and with modulus one.

Finally, these ~~ideas will lead, as it will be shown, to the~~ solution ~~of one of problems posed at the end of [28]~~

Throughout the present work, the notation of [28] will be preserved except for the following: a sequence of integers will be denoted $\theta = \{\theta(i)\}_1^\infty$, $\mathcal{Q}_N(T)$ will be replaced by $\mathcal{Q}_\theta(T)$, $\mathcal{Q}(T)$, $\alpha_N(T)$, $\alpha(T)$ will be replaced by $\mathcal{Q}(T)$, $\alpha_\theta(T)$ and $\alpha(T)$ respectively.

§1. The σ-algebras $\mathcal{Q}_\theta(T)$.

Definition 1.1.

Let T be an invertible measure preserving transformation on a Lebesgue space $(X,\mathcal{B},m)$. Let $\theta = \{\theta(i)\}$ be a sequence of integers, then $\mathcal{Q}_\theta(T)$ is defined by

$$\mathcal{Q}_\theta(T) = \{A \in \mathcal{B} \mid m(T^{\theta(i)} A \,\Delta\, T^{\theta(i)} A) \to 0 \text{ as } i,j \to \infty\}$$

Theorem 1.2.

$\mathcal{Q}_{\theta}(T)$ is a σ-algebra

Proof.

We have to prove: i) $\emptyset \in \mathcal{Q}_{\theta}(T)$, ii) $\mathcal{Q}_{\theta}(T)$ is closed under complementation, iii) $\mathcal{Q}_{\theta}(T)$ is finitely additive, and iv) $\mathcal{Q}_{\theta}(T)$ is σ-additive.

Clearly i) is true. As for ii), if $A \in {}_{\theta}(T)$ then $m(T^{\theta(i)}A^c \Delta T^{\theta(j)}A^c) = m(T^{\theta(i)}A \Delta T^{\theta(j)}A)$ which implies that $A^c \in \mathcal{Q}_{\theta}(T)$. Let $A_1, A_2 \in \mathcal{Q}_{\theta}(T)$ then $m(T^{\theta(i)}(A_1 \cup A_2) \Delta T^{\theta(j)}(A_1 \cup A_2)) \leqslant m(T^{\theta(i)}A_1 \Delta T^{\theta(j)}A_1) + m(T^{\theta(i)}A_2 \Delta T^{\theta(j)}A_2) \longrightarrow 0$

Thus iii) is true. Finally to prove that $\mathcal{Q}_{\theta}(T)$ is σ-additive it is enough to prove that if $(A_k)_{k\geqslant 1} \subset \mathcal{Q}_{\theta}(T)$ and $A_1 \subset A_2 \subset A_3 \subset \ldots$. then $A = \bigcup_{k\geqslant 1} A_i \in \mathcal{Q}_{\theta}(T)$.

Given $\varepsilon > 0$, choose k_o s.t. $m(A - A_{k_o}) < \varepsilon$. Let I be chosen s.t. $i,j > I \Rightarrow m(T^{\theta(i)} A_{k_o} \Delta T^{\theta(j)} A_{k_o}) < \varepsilon$.

Then

$$
\begin{aligned}
m(T^{\theta(i)}A \Delta T^{\theta(j)}A) &\leqslant m(T^{\theta(i)}A \Delta T^{\theta(i)}A_{k_o}) + m(T^{\theta(i)}A_{k_o} \Delta T^{\theta(j)}A_{k_o}) \\
&\quad + m(T^{\theta(j)}A_{k_o} \Delta T^{\theta(j)}A) \\
&< 3\varepsilon.
\end{aligned}
$$

Therefore $A \in \mathcal{Q}_{\theta}(T)$ and iv) is true, and this concludes the proof that $\mathcal{Q}_{\theta}(T)$ is a σ-algebra.

Corollary 1.3.

Let $\mathcal{A}'_\theta(T)$ be Walter's σ-algebra, then $\mathcal{A}'_\theta(T) \subset \mathcal{A}_\theta(T)$.

Proof.

If $A \in \mathcal{A}'_\theta(T)$ then

$m(T^{\theta(i)} A \,\Delta\, T^{\theta(j)}A) \leqslant m(T^{\theta(i)}A \,\Delta\, A) + m(A \,\Delta\, T^{\theta(j)}A) \to 0$, i.e. $A \in \mathcal{A}_\theta(T)$

and hence $\mathcal{A}'_\theta(T) \subset \mathcal{A}_\theta(T)$.

Lemma 1.4.

Let $f \in L^2(\mathcal{B})$ be real valued and non-constant. Let θ be a sequence of integers and $\mathcal{C}$ denotes the σ-algebra of Borel sets of $\mathbb{R}$. If $\|U_T^{\theta(i)}f - U_T^{\theta(j)}f\| \to 0$, then $f^{-1}(\mathcal{C}) \subset \mathcal{A}_\theta(T)$.

Proof.

Let $b \in \mathbb{R}$ and put $B = \{x \mid f(x) \leqslant b\}$ and $B_\varepsilon = \{x \mid f(x) \leqslant b + \varepsilon\}$. On the set $T^{-\theta(i)}B - T^{-\theta(j)}B_\varepsilon$ we have $|f(T^{\theta(i)}x) - f(T^{\theta(j)}x)| \geqslant \varepsilon$, (since: $x \in T^{-\theta(i)}B - T^{-\theta(j)}B_\varepsilon \Rightarrow x = T^{-\theta(i)}y$, $y \in B \Rightarrow T^{\theta(i)}x = y$ and $f(y) \leqslant \varepsilon$; now $T^{-\theta(j)}T^{-\theta(i)}y \notin B_\varepsilon$, otherwise $T^{\theta(j)}y \in T^{-\theta(i)}y \in B_\varepsilon$ a contradiction).

Therefore $m(T^{-\theta(i)}B - T^{-\theta(j)}B_\varepsilon) \to 0$. Since $B_\varepsilon - B$ decrease with ε, then $\bigcap_{\varepsilon>0}(B_\varepsilon - B) = \emptyset$.

Given $\delta > 0$ choose ε_0 such that $m(B_{\varepsilon_0} - B) < \delta$. Choose I such that

$$i,j > I \Rightarrow m(T^{-\theta(i)}B \setminus T^{-\theta(j)}B) \leqslant m(T^{-\theta(i)}B - T^{-\theta(j)}B_{\varepsilon_0}) + m(T^{-\theta(j)}B_{\varepsilon_0} - T^{-\theta(j)}B) < 2\delta.$$

Therefore $m(T^{\theta(i)}B \,\Delta\, T^{\theta(j)}B) \to 0$, which implies that $f^{-1}(-\infty, b) \in \mathcal{A}_\theta \subset T)$.

Thus $f^{-1}(\mathcal{C}) \subset \mathcal{A}_\theta(T)$.

Theorem 1.5.

$$L^2(\mathcal{C}_\theta(T)) = \{f \in L^2 \subset \mathcal{B} \mid \| U_T^{\theta(i)}f - U_T^{\theta(j)}f \| \to 0\}$$

Proof.

The proof goes word by word as the proof of theorem (2) in [28], but with replacing $\mathcal{C}_N$ by $\mathcal{C}_\theta$.

Let W be the space of invertible measure-preserving transformations of $(X, \mathcal{B}, m)$ endowed with the weak topology [10]. It has been stated in [28] that $\mathcal{C}'_\theta(T) = \mathcal{B}$ means $(T^{\theta(i)})$, converges to the identity in W or equivalently $(U_T^{\theta(i)})$ converges to I in the space of unitary operators in $L^2(\mathcal{B})$ with the weak (strong) topology. In the next theorem we will prove that $\mathcal{C}_\theta(T) = \mathcal{B}$ means $(T^{\theta(i)})$ converges to an invertible measure preserving transformation S in W. It is sufficient to prove that $U_T^{\theta(i)}$ converges to V in the space of unitary operators in $L^2(\mathcal{B})$ with the weak (strong) topology, and that V is unitary and induced i.e. $V = V_S$ for some S in W.

Theorem 1.6.

$$\mathcal{C}_\theta(T) = \mathcal{B} \Rightarrow U_T^{\theta(i)} \xrightarrow{S} V \text{ and V is unitary and induced.}$$

Proof.

$$\mathcal{C}_\theta(T) = \mathcal{B} \Rightarrow \| U_T^{\theta(i)}f - U_T^{\theta(j)}f \| \to 0, \forall f \in L^2(\mathcal{B}) \text{ and}$$

this implies that $(U_T^{\theta(i)})$ converges strongly to V in the space of bounded linear operators in $L^2(\mathcal{B})$. That V is unitary follows at once from the computation of $\| U_T^{\theta(i)}f - V f\|$ which tends to zero. To prove that V is induced it remains to prove that V is multiplicative. It suffices to prove that for bounded functions. Let f, g be essentially bounded.

Hence,

$$\| U_T^{\theta(i)}f - Vf\| \to 0, \quad \| U_T^{\theta(i)}g - Vg\| \to 0 \text{ and } \| U_T^{\theta(i)}(fg) - V(fg)\| \to 0.$$

We want to prove that $V(fg) = Vf\ Vg$ or $\| U_T^{\theta(i)}(fg) - Vf.Vg\| \to 0$. But

$$\| U_T^{\theta(i)}(fg) - Vf\ Vg\| = \| U_T^{\theta(i)}f\ U_T^{\theta(i)}g - U_T^{\theta(i)}g.Vf + U_T^{\theta(i)}gVf - VfVg\|$$

$$\leq \| U_T^{\theta(i)}f\ U_T^{\theta(i)}g - U_T^{\theta(i)}gVf\| + \| U_T^{\theta(i)}gVf - Vf\ Vg\|$$

$$\leq \|g\|_\infty \| U_T^{\theta(i)}f - Vf\| + \|f\|_\infty \| U_T^{\theta(i)}g - Vg\|$$

$$\to 0.$$

Thus V is unitary and multiplicative i.e. $V = V_S$ for some $S \in W$.

The next theorem relates the property $\mathcal{a}_\theta(T) = \mathcal{B}$ to the spectral theory.

<u>Theorem 1.7.</u>

$$\mathcal{a}_\theta(T) = \mathcal{B} \iff \int_K |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\sigma(\lambda) \to 0 \text{ as } i,j \to \infty, \text{ where}$$

σ denotes a finite measure on $K = \{z \in \mathbb{C} \mid |z| = 1\}$ whose type is a maximal spectral type of T.

<u>Proof</u>.

Suppose the right hand side holds. We want to prove that this implies the same holds for σ_f where σ_f is the maximal spectral type of f, $f \in L^2(\mathcal{B})$. Let $h \in L^1(\sigma)$. Let $\delta > 0$ be given and choose h_1, h_2 such that $h = h_1 + h_2$, h_1 is bounded ($|h_1(\lambda)| < C_\delta$ say) and $\int |h_2(\lambda)|d\sigma(\lambda) < \delta$.

Then,

$$\left|\int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 |h(\lambda)d\sigma(\lambda)|\right| \leq \int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 |h_1(\lambda)|d\sigma$$

$$+ \int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 |h_2(\lambda)|^2 d\sigma$$

$$< c_\delta \int |\lambda^{\theta(i)} - \lambda^{\theta(j)}| d\sigma(\lambda) + 4\delta < 5\delta \text{ if } i,j > I, I$$

is choosen such that $i, j > I \Longrightarrow \int |\lambda^{\theta(j)} - \lambda^{\theta(j)}|^2 d\sigma < \delta$.

If h is the Radon-Ninkodym derivative of σ_f w.r.t. σ, then following theorem 1.5 we have

$$\| U_T^{\theta(i)}f - U_T^{\theta(j)}f\| \to 0, \ \forall f \in L^2 \subset \mathcal{B} \text{ i.e. } \mathcal{Q}_\theta(T) = \mathcal{B}.$$

Conversely, $\mathcal{Q}_\theta(T) = \mathcal{B} \Longrightarrow \int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d_f \to 0$. In particular if f is maximal i.e.

$$\int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\sigma \to 0, \ \sigma \text{ is of maximal spectral type.}$$

§2. Some properties of $\mathcal{Q}_\theta(T)$.

The following theorem which gives the simplest examples of transformations with $\mathcal{Q}_\theta(T) = \mathcal{B}$, has been mentioned in [28]; we will write it down for completeness sake.

Theorem 2.1. [28]

If T is ergodic with discrete spectrum, there exist a sequence $\theta = \{\theta(i)\}$ with $\mathcal{Q}'_\theta(T) = \mathcal{B}$.

Proof.

We can suppose T is an ergodic rotation on a compact abelian group G [10, p.48]. Choose $\theta = \{\theta(i)\}$ such that $a^{\theta(i)} \to e$ the identity element of G. If γ is a character of G $\| U_T^{\theta(i)}\gamma - \gamma \|^2 = |\gamma(a^{\theta(i)}) - 1|^2 \to 0$. Since the characters generate $L^2(G)$ we have $\mathcal{A}'_\theta(T) = \mathcal{B}$.

It follows from theorem (2.1) and Corollary (1.3) that ergodic transformations with discrete spectrum are the simplest examples of transformations with $\mathcal{A}_\theta(T) = \mathcal{B}$. Also, it follows from Corollary 1.3 that the algebras $\mathcal{A}_\theta(T)$ has the same connection, with the work of Katok and Stepin [12] and the work of Chacon and Schwartzbauer [1] on approximation by periodic transformation, as the algebras $\mathcal{A}'_\theta(T)$. If T admits an approximation of the second kind by periodic transformation (a.p.t.II) in the sense of Katok and Stepin [12,p.78] then $\mathcal{A}'_{\{P_i\}} = \mathcal{B}$ and then $\mathcal{A}_{\{P_i\}} = \mathcal{B}$ as well by Corollary 1.3. Also if T admits an approximation by periodic automorphisms in Chacon and Schwartzbauer, sense [1] then $\mathcal{A}'_{\{q_i\}} = \mathcal{B}$ and then $\mathcal{A}_{\{q_i\}} = \mathcal{B}$ by Corollary 1.3.

The relationship of the partitions $\alpha'_\theta(T)$ ($\alpha'_\theta(T)$ is the partition corresponding to the σ-algebra $\mathcal{A}'_\theta(T)$ to the entropy theory has been discussed in [28]. Here, the relationship of the partitions $\alpha_\theta(T)$ ($\alpha_\theta(T)$ is the partition corresponding to the σ-algebra $\mathcal{A}_\theta(T)$) to entropy theory will be discussed as well. It will turn out that the relationship is similar, however, the proofs will be a modified version of the ones in [28].

The following is a summary of some relevant results in entropy theory and is quoted from [28]. (The notation for entropy are from [25]).

"Let $\mathfrak{X}$ denote the set of partitions with finite entropy. Pinsker [21] has defined the maximum partition with zero entropy for T as $\Pi(T) = V\{\xi \in \mathfrak{X} | h(T,\xi) = 0\}$. $\Pi(T)$ is invariant with respect to T and if $\xi \in \mathfrak{X}$ then $\xi \leq \Pi(T)$ iff $h(T, \xi) = 0$. Using the concept of sequence entropy introduced by Kushnirenko [13], one can define the maximum partition with zero θ-entropy for T (i.e. for a sequence of integers θ) by $\Pi_\theta(T) = \{\xi \in \mathfrak{X} | h_\theta(T,\xi) = 0\}$. It is straightforward to check that $T\,\Pi_\theta(T) = \Pi_\theta(T)$ and if $\xi \in \mathfrak{X}$ then $\xi \leq \Pi_\theta(T)$ iff $h_\theta(T)\xi) = 0$. The main result of Kushnirenko's paper [13] implies $\Pi_\infty(T) = \bigwedge_\theta \Pi_\theta(T)$ is the maximum partition for T such that the associated factor transformation has discrete spectrum. In other words $\Pi_\infty(T)$ is the partition generated by the eigen functions of T."

<u>Theorem 2.2</u>.

For every sequence θ of integers,

i) $\alpha_\theta(T) \leq \Pi_\theta(T)$,

ii) $\alpha_\theta(T) \leq \Pi(T)$, and

iii) $\alpha(T) \leq \Pi(T)$

<u>Proof</u>

i) Let ξ be a finite partition and $\xi \leq \alpha_\theta(T)$, then

$$H(T^{\theta(1)}\xi V \ldots\ldots VT^{\theta(k)}\xi) \leq H(T^{\theta(1)}\xi) + H(T^{\theta(2)}\xi | T^{\theta(1)}) + \ldots\ldots$$
$$\ldots\ldots + H(T^{\theta(k)}\xi | T^{\theta(k-1)}\xi),$$

So if $H(T^{\theta(k)}\xi | T^{\theta(k-1)}\xi) \to 0$ then $h_\theta(T,\xi) = 0$ and $\xi \leq \Pi_\theta(T)$. But, $H(T^{\theta(k)}\xi | T^{\theta(k-1)}) \leq H(T^{\theta(k)}\xi | S\xi) + H(S\xi | T^{\theta(k-1)}\xi)$ where $S = \lim_k T^{\theta(k)}$ in the group of invertible measure preserving transformation on $(X, \mathcal{Q}_\theta, m)$, so $H(T^{\theta(k)}\xi | T^{\theta(k-1)}\xi) \to 0$ as $k \to 0$ since the right hand side of the above equality does. Hence $\alpha_\theta(T) \leq \Pi_\theta(T)$.

ii) Let ξ be a finite partition and $\xi < \alpha_\theta(T)$, then
$h(T,\xi) = H(\xi| \bigvee_{n=1}^{\infty} T^n\xi) < \lim_{i\to\infty} H(\xi|T^{\theta(i)-\theta(i-1)}\xi) \to 0)$, and then ii) follows.

iii) i) and ii) imply ii).

<u>Corollary 2.3.</u>

$$h(T_{\alpha_\theta(T)}) = 0, \ h_\theta(T_{\alpha_\theta(T)}) = 0 \text{ and } h(T_{\alpha(T)}) = 0$$

Let $\mathcal{C}_\theta$ be the class of measure preserving transformations with $\mathcal{A}_\theta = \mathcal{B}$. If $\mathcal{C}'_\theta$ is the class of all measure preserving transformation with $\mathcal{A}'_\theta = \mathcal{B}$ then, clearly, $\mathcal{C}'_\theta \subset \mathcal{C}_\theta$. It has been proved in [28] that $\mathcal{C}'_\theta$ is closed under reasonable finite operations. It will be proven, here, that $\mathcal{C}_\theta$ is closed under such operations as well. Moreover, following Parry's idea [19] of a structure theory based on models we will introduce the concept of dynamical representations in $\mathcal{C}_\theta$.

<u>Lemma 2.4.</u>

If $S, T \in \mathcal{C}_\theta$ then
$$\mathcal{A}_\theta(S \times T) = \mathcal{A}_\theta(S) \times \mathcal{A}_\theta(T).$$

<u>Proof.</u>

Let $R = \{A_1 \times A_2, A_1 \in \mathcal{A}_\theta(S), A_2 \in \mathcal{A}_\theta(T)\}$, the set of measurable rectangles in $\mathcal{A}_\theta(S) \times \mathcal{A}_\theta(T)$. R is an algebra which generates $\mathcal{A}_\theta(S) \times \mathcal{A}_\theta(T)$. It is sufficient to prove that R lies in $\mathcal{A}_\theta(S \times T)$. Let $A_1 \in \mathcal{A}_\theta(S)$ and $A_2 \in \mathcal{A}_\theta(T)$, then

$$(m_1 \times m_2)\ [\{(S\times T)^{\theta(i)}(A_1 \times A_2)\}\ \Delta\ \{(S\times T)^{\theta(j)}(A_1 \times A_2)\}]$$

$$= (m_1 \times m_2)\ [\{(S^{\theta(i)}A_1\ \Delta\ S^{\theta(j)}A_1 \times (T^{\theta(i)}A_2\ \Delta\ T^{\theta(j)}A_2)\}$$

$$\cup\{(S^{\theta(i)}A_1 \cap S^{\theta(j)}A_1) \times (T^{\theta(j)}A_2\ \Delta\ T^{\theta(j)}A_2)\}$$

$$\cup\{(S^{\theta(i)}A_1\ \Delta\ S^{\theta(j)}A_1) \times (T^{\theta i)}A_2 \cap T^{\theta(j)}A_2)\}$$

$$= m_1(S^{\theta(i)}A_1\ \Delta\ S^{\theta(j)}A_2).\ m_2(T^{\theta(i)}A_2\ \Delta\ T^{\theta(j)}A_2)$$

$$+ m_1(S^{\theta(i)}A_1 \cap S^{\theta(j)}A_1).\ m_2(T^{\theta(i)}A_2\ \Delta\ T^{\theta(j)}A_2)$$

$$+ m_1(S^{\theta(i)}A_1\ \Delta\ S^{\theta(j)}A_1).\ m_2(T^{\theta(i)}A_2 \cap T^{\theta(j)}A_2)$$

$$\to 0 \text{ as } i,\ j \to 0.$$

Thus $R \subset \mathcal{Q}_\theta(S \times T)$ and

$$\mathcal{Q}_\theta(S \times T) = \mathcal{Q}_\theta(S) \times \mathcal{Q}_\theta(T).$$

The last lemma is, in fact, valid for countable cartesian products and the proof is similar.

Lemma 2.5.

$$\mathcal{Q}(S \times T) = \mathcal{Q}(S) \times \mathcal{Q}(T)$$

Proof.

From lemma 2.4. we have $\mathcal{Q}_\theta(S \times T) = \mathcal{Q}_\theta(S) \times \mathcal{Q}_\theta(T)$. The results then follow since the cartesian product preserves the refinement over θ.

Lemma 2.6.

$\mathcal{C}_\theta$ is closed under inverse limits and if $\xi_n \nearrow \xi$ and $T\,\xi_n = \xi_n$, $T\xi = \xi$, then $\alpha_\theta(T_{\xi_n}) \nearrow \alpha_\theta(T_\xi)$

Proof.

The second statement implies the first. To prove the second statement it suffices to take $\xi = \varepsilon$ (ε is the partition of the space into points) If $f \in L^2(\mathcal{Q}_\theta(T)$ and $f_n = E(f|\xi_n)$, where $E(f|\xi_n)$ is the conditional expectation of f relative to the σ-algebra generated by ξ_n, then

$$\| f - f_n \| \to 0 \text{ and } \| U^{\theta(i)}_{T_{\xi_n}} f_n - U^{\theta(j)}_{T_{\xi_n}} f_n \|$$

$$\leq \| U^{\theta(i)}_T f - U^{\theta(j)}_T f \| \to 0. \text{ Thus } f_n \in L^2(\alpha_\theta(T_{\xi_n})) \text{ and}$$

$$\alpha_\theta(T_{\xi_n}) \to \alpha_\theta(T_\xi).$$

Definition 2.7.

i) Let $(X, \mathcal{B}, m, T)$ and $(X', \mathcal{B}', m', T')$ be two dynamical systems. Then $(X', \mathcal{B}', m', T')$ is a factor of $(X, \mathcal{B}, m, T)$ if there exist a homomorphism $\emptyset$ of $(X, \mathcal{B}, m)$ onto $(X', \mathcal{B}', m')$ such that

$$\emptyset T = T' \emptyset$$

ii) Let $(X, \mathcal{B}, m, T)$ be a dynamical system and $(X, \mathcal{B}', m', T')$ a dynamical system with $\mathcal{Q}_\theta(T') = \mathcal{B}$ i.e. $T' \in \mathcal{C}_\theta$. Then a homomorphism $\emptyset_\alpha$ of $(X, \mathcal{B}, m)$ onto $(X', \mathcal{B}', m)$ is said to be a representation of T in $\mathcal{C}_\theta$ if $\emptyset_\alpha T = T' \emptyset_\alpha$.

$\mathcal{C}_\theta$ is closed under factors. Moreover the relation "$\leq$" in $\mathcal{C}_\theta$, where $S_1 \leq S_2$ means S_1 is a factor of S_2, $S_1, S_2 \in \mathcal{C}_\theta$, is a partial order. Since S_1, and S_2 in $\mathcal{C}_\theta$, imply that $S_1 \times S_2 \in \mathcal{C}_\theta$, then $\mathcal{C}_\theta$ is, in fact, a directed set by "$\leq$".

Definition 2.8.

Let $\mathcal{B}$ and $\mathcal{B}_\alpha$ are the σ-algebras associated with T and T_α respectively. Then T is said to have sufficiently many representation in $\mathcal{C}_\theta$ if $\mathcal{B} = \bigvee_\alpha \phi_\alpha^{-1}(\mathcal{B}_\alpha)$, ϕ_α is a representation of T in $\mathcal{C}_\theta$.

Let T be ergodic with representations ϕ_α and ϕ_β in T_α, $T_\beta \in \mathcal{C}_\theta$ respectively. Then there exist $T_{\alpha\vee\beta}$ in $\mathcal{C}_\theta$ and homomorphisms $\phi_{\alpha\vee\beta}$, ϕ'_α and ϕ'_β such that the following diagram commutes (Note: a dynamical system $(X_\alpha, \mathcal{B}_\alpha, m_\alpha, T_\alpha)$ will be abbreviated T_α)

$$\begin{tikzcd}
& T \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] \arrow[dd] & \\
T_\alpha & & T_\beta \\
& T_{\alpha\vee\beta} \arrow[ul, "\phi'_\alpha"] \arrow[ur, "\phi'_\beta"'] &
\end{tikzcd}$$

$\phi_{\alpha\vee\beta}$ is the representation of T in $T_{\alpha\vee\beta}$ in $\mathcal{C}_\theta$. The justification for existance follow from the construction of the class $\mathcal{C}_\theta$, it is in one-to-one correspondence with its associated σ-algebras. Also that construction implies that $T_{\alpha\vee\beta}$ is unique.

Now we are in a position to give the following theorem.

Theorem 2.9.

If T is ergodic with sufficiently many representations in $\mathcal{C}_\theta$, then T is metrically isomorphic to an inverse limit of a sequence of transformation in $\mathcal{C}_\theta$ and that $T \in \mathcal{C}_\theta$.

Proof.

Let $\mathcal{T}$ be the set of transformation representing T in $\ _\theta$. As it has been mentioned above, if $T_\alpha, T_\beta \in \mathcal{T}$ then $T_{\alpha\vee\beta}$ is well defined. Therefore $\mathcal{T}$ is a directed family by "$\leqslant$". Also $\ = \bigcup_{\alpha\in I} \phi_\alpha^{-1}(\ _\alpha)$ is a sub-σ-algebra generates $\mathcal{B}$. Since $\mathcal{B}$ is separable, there is a dense sequence $\{B_n\}$ in $\mathcal{B}$. We can inductively choose $\alpha_n \geqslant \alpha_{n-1} \geqslant \ldots\ldots \geqslant \alpha_1$ so that $B_1, \ldots\ldots, B_n$ can be approximated to within $\frac{1}{n}$ by sets in $\phi_{\alpha_n}^{-1}(\ _{\alpha_n})$. If we write n instead of α_n then $\phi_n^{-1}(\ _n) \supset \phi_{n-1}^{-1}(\ _{n-1})$ and $\bigcup_n \phi_n^{-1}(\ _n)$ generates . Moreover we have the homomorphisms π_n with $T_n \xrightarrow{\pi_n} T_{n-1}$. The inverse limit

$\varprojlim (X_n, T_n) = (X_\infty, T_\infty)$ is defined

$$X_\infty = \{(x_1, x_2, \ldots\ldots) : \pi_n(x_n) = x_{n-1}\}$$

$$\subset X_1 \times X_2 \times \ldots\ldots\ldots$$

and $T_\infty(x_1, x_2, \ldots\ldots) = (T_1x,\ T_2x\ldots\ldots)$. It is well known that T is metrically isomorphic to T_∞. (The idea of dynamical representation in $\ _\theta$ is substantially influenced by the ideas and proofs in Parry's work [19], entitled dynamical representation in nil-manifolds.

Theorem 2.10.

The σ-algebra $\bigcap_\theta \mathcal{C}_\theta(T)$ is the sub-σ-algebra of T invariant sets.

Proof. Let $A \in \bigcap_\theta \mathcal{C}_\theta(T)$ then

$$
\begin{aligned}
m(TA \Delta A) &= m(T^{n-m+1}A \Delta T^{n-m}A) \\
&\le m(T^{n-m+1}A \Delta A) + m(T^{n-m}A \Delta A) \\
&= m(T^{n-1}A \Delta T^{m}) + m(T^{n} A \Delta T^{m}A) \\
&\to 0 \qquad \text{as } n, m \to \infty.
\end{aligned}
$$

§3 T-groups.

Let W be the class of invertible measure-preserving transformation on a Lebesgue space $(X, \mathcal{B}, m)$. W is a group with composition of map as the group operation. Let $\{E_n\}$ be a countable dense set in $\mathcal{B}$ ($\mathcal{B}$ (mod sets of measure zero) a complete separable metric space with metric $\mathcal{D}$, where $\mathcal{D}(A,B) = m(A \Delta B)$, for all $A, B \in \mathcal{B}$). Define the distance between S and T in W as

$$d(S,T) = \sum_m \frac{1}{2^n} (m(SE_n \Delta TE_n) + m(S^{-1} E_n \Delta T^{-1}E_n)).$$

The metric induces the weak topology described in the introduction to the present chapter. So, (W,d) is a complete metric group see [10] for details.

Definition 3.1.

The T-set $G(T)$ associated with an invertible measure preserving transformation $T \in W$ is a subset of W defined as follows

$$G(T)=\{S \in W \mid T^{\theta(i)} \xrightarrow{w} S \text{ for some sequence } \theta \text{ of integers}\} \cup \{T^n, n \in Z\},$$

where W means weak convergence and T^0 is, by definition the identity transformation. It should be remarked that according to theorem (1.6) $T^{\theta(i)} \xrightarrow{w} S$ is equivalent to $\mathcal{C}_\theta(T) = \mathcal{B}$.

Theorem 3.2.

$G(T)$ is a commutative subgroup of W.

Proof.

The proofs will be divided into two parts:

A) G(T) is closed under multiplication and multiplication is commutative.

If S and S' are powers of T then $SS' \in G(T)$. Let

$T^{\theta(i)} \xrightarrow{w} S$ i.e. $m(T^{\theta(i)} A \Delta SA) \to 0$ for all $A \in \mathcal{B}$, and $S = T^k$

for some integer k. If $\theta' = \{\theta(i) + k\}$, then $T^{\theta'(i)} \to ST^k$ since $\mathcal{C}_{\theta'}(T) = \mathcal{B}$. Thus $ST^k \in G(T)$. That $ST = ST$ is clear and hence $ST^k = T^kS$. Let $T^{\theta_1(i)} \xrightarrow{w} S$ and $T^{\theta_2(i)} \to S_2$. Then if $\theta'=\{\theta_1(i)+\theta_2(i)\}$, we have $T^{\theta(i)} \to S_1S_2$ since

$$m(T^{\theta_1(i)} T^{\theta_2(i)} A \Delta S_1S_2 A)$$

$$\leq m(T^{\theta_1(i)} T^{\theta_2(i)} A \Delta T^{\theta_1(i)} S_2A)$$

$$+ m(T^{\theta_1(i)} S_2 A \Delta S_1S_2 A) \to 0,$$ and thus $\mathcal{C}_{\theta'}(T) = \mathcal{B}$. Also $S_1S_2 = S_2S_1$ is clear. Thus A is proved.

B) <u>If $S \in G(T)$ then $S^{-1} \in G(T)$.</u>

Let $S = T^k$ then $t^{-k} \in G(T)$ by definition. If $T^{\theta(i)} \to S$ then $T^{-\theta(i)} \to S$ and $T^{-\theta(i)} \to S^{-1}$ since

$$m(T^{-\theta(i)}A \,\Delta\, S^{-1}A) = m(ST^{-\theta(i)}A \,\Delta\, A)$$

$$= m(SA \,\Delta\, T^{\theta(i)}A) \to 0.$$

Thus B is proved.

A) and B) imply that the theorem is true.

<u>Lemma 3.3.</u>

$$G(T) = G(T^{-1})$$

<u>Proof.</u>

Obvious.

<u>Theorem 3.4.</u>

G(T) is a metric invariant for T.

<u>Proof.</u>

If T_1 is conjugate to T_2, then there exist a $w \in W$ such $w^{-1}T_2W_2 = T_1$. Let $S \in G(T_1)$ i.e. there exist a sequence θ of integers such that $T_1{}^{\theta(i)} \xrightarrow{W} S$. Then the equality

$$m((W^{-1}T_2W)^{\theta(i)}A \,\Delta\, SA) = m(T_2{}^{\theta(i)}A \,\Delta\, (WSW^{-1})A) \text{ implies that}$$

$T_2{}^{\theta(i)} \xrightarrow{W} S'$ and S' is conjugate to S. The converse is true as well and therefore G(T) is a metric invariant for T.

G(T) endowed with the induced weak topology from W is a topological group. Since G(T) is commutative, the metric d restricted to G(T) is invariant.

§4. Mixing Properties.

In this section, the relation of the σ-algebras $\mathcal{Q}_\theta$ with mixing concepts will be discussed. It will turn out that the relation has the same features as that of the σ-algebras $\mathcal{Q}'_\theta$. The most important results of this section is: if $\mathcal{Q}_\theta(T) = \mathcal{B}$ then T is disjoint from all strong mixing transformation. (For the definition of disjointness see [5]).

Theorem 4.1.

There are no non-constant mixing functions in $L^2(\mathcal{Q}_\theta(T))$ (i.e. $(U_T^n f, f) \to (f,1)(1,f)$ for $f \in L^2(\mathcal{Q}_\theta(T))$ implies f = constant)

Proof

If $f \in L^2(\mathcal{Q}'_{\bar{\bar{\theta}}}(T))$ for some sequence integers $\bar{\bar{\theta}}$ ($\mathcal{Q}'_{\bar{\bar{\theta}}}(T)$ is Walters' algebra) and $(U_T^n f, f) \to (f,1)(1,f)$ then f = constant (Theorem 7 [28]).

Let $f \in L^2(\mathcal{Q}_\theta(T))$ and $(U_T^n f, f) \to (f,1)(1,f)$.

Given the sequence $\theta = \{\theta(i)\}$ one can find a subsequence $\bar{\theta}$ of θ for which $\bar{\theta}(i) - \bar{\theta}(i-1) \to \infty$.

Let $\bar{\bar{\theta}}(i) = \bar{\theta}(i) - \bar{\theta}(i-1)$. Now $f \in L^2(\mathcal{Q}_\theta(T)) \Rightarrow f \in L^2(\mathcal{Q}'_{\bar{\bar{\theta}}}(T))$

Since

$$\| U_T^{\bar{\bar{\theta}}(i)} f - f \| = \| U_T^{\bar{\theta}(i) - \bar{\theta}(i-1)} f - f \|$$
$$< \| U_T^{\bar{\theta}(i)} f - U_S f \| + \| U_S f - U_T^{\bar{\theta}(i-1)} f \|$$
$$\to 0 \text{ where } S = \lim T^{\theta(i)} .$$

But $f \in L^2(\mathcal{Q}'_{\bar{\bar{\theta}}}(T))$ and $(U_T^n f, f) \to (f,1)(1,f)$ implies f = constant.

Thus the theorem is proved.

Corollary 4.2.

$T_{\alpha_\theta(T)}$ has singular spectrum

Proof

Suppose that this is not true, then there exists $f \in L^2(\mathcal{A}_\theta(T))$ with absolutely continuous spectral me-sure σ_f. Then $(U_T^n f, f) = \int \lambda^n \, d\sigma_f \to 0$ by Riemann Lebesgue lemma. But $f \in L^2(\mathcal{A}_\theta(T))$ implies $f \in L^2(\mathcal{A}'_{\overline{\overline{\theta}}}\ (T))$ (see Theorem 4.1) and $f \in L^2\ (L^2(\mathcal{A}'_{\overline{\overline{\theta}}}\ (T))$ implies $(U_T^{\theta(i)} f, f) \to \|f\|^2$, therefore $f = 0$.

Corollary 4.3.

If T is totally ergodic with quasi discrete spectrum then $\alpha_\theta(T) < \Pi_\infty(T)$ for all sequences θ and there exist a sequence θ with $\alpha_\theta(T) = \Pi_\infty(T)$. Hence $\alpha(T) = \Pi_\infty(T)$.

Proof.

Let $L^2(\varepsilon) = L^2(\Pi_{\infty}(T)) \oplus \mathcal{H}$ where $U_T|\mathcal{H} =$ and $U_T|\mathcal{H}$ has Lebesgue spectrum. Then if $f \in L^2(\alpha_\theta(T))$, f can be written as $f = f_1 + f_2$ where $f_1 \in L^2(\Pi_{\infty}(T))$ and $f_2 \in \mathcal{H}$. Thus

$$\| U_T^{\theta(i)}f - U_T^{\theta(j)}f\| = \| U_T^{\theta(i)}f_1 - U_T^{\theta(j)}f_1\|^2 + \| U_T^{\theta(i)}f_2 - U_T^{\theta(j)}f_2\|$$

implies $f_1 \in L^2(\alpha_\theta(T))$ and $f_2 \in L^2(\alpha_\theta(T))$. By Corollary 4.3. $f_2 = 0$ and hence $\alpha_\theta(T) < \Pi_{\infty}(T)$. The rest of this Corollary follows from (2.1).

In the following, we shall investigate the relation of the algebras $\mathcal{Q}_\theta(T)$ with various mixing concepts. But first, the concept of intermixing will be recalled.

Definition 4.4.

T is intermixing if whenever $m(A) > 0$ and $m(B) > 0$ and $A,B \in \mathcal{B}$, then

$$\liminf_{n\to} m(T^n A \cap B) > 0.$$

Note: Friedmann and Ornstein [4], give examples of intermixing transformation which are not strong mixing.

Theorem 4.5.

T strong mixing $\overset{\Rightarrow}{\not\Leftarrow}$ T intermixing $\Rightarrow \mathcal{Q}(T) = \mathcal{N}$ (the trivial σ-algebra) $\underset{\not\Leftarrow}{\Rightarrow}$ T weak mixing.

Proof.

That T is strong mixing implies T is intermixing is clear. Friedmann and Ornestein [ ] has shown that the converse is false. If T is intermixing and $0 < m(A) < 1$, $A \in \mathcal{B}$, then

$$\liminf_{n\to\infty} m(T^n A \cap (SA)^c) > 0$$ for any transformation $S \in W$ and so, $A \notin \mathcal{Q}_\theta(T)$ for any sequence θ. Thus $\mathcal{Q}(\theta) = \mathcal{N}$. If T is ergodic with discrete spectrum then there exist a sequence θ such that $\mathcal{Q}_\theta(T) = \mathcal{B}$ and therefore $\mathcal{Q}(T) = \mathcal{N}$ implies T is weak mixing and the converse is false by examples of § 6 .

Proof.

Let $L^2(\varepsilon) = L^2(\Pi_\infty(T)) \oplus \mathcal{H}$ where $U_T|\mathcal{H} = \mathcal{H}$ and $U_T|\mathcal{H}$ has Lebesgue spectrum. Then if $f \in L^2(\alpha_\theta(T))$, f can be written as $f = f_1 + f_2$ where $f_1 \in L^2(\Pi_\infty(T))$ and $f_2 \in \mathcal{H}$. Thus

$$\| U_T^{\theta(i)}f - U_T^{\theta(j)}f\| = \| U_T^{\theta(i)}f_1 - U_T^{\theta(j)}f_1\|^2 + \| U_T^{\theta(i)}f_2 - U_T^{\theta(j)}f_2\|$$

implies $f_1 \in L^2(\alpha_\theta(T))$ and $f_2 \in L^2(\alpha_\theta(T))$. By Corollary 4.3. $f_2 = 0$ and hence $\alpha_\theta(T) < \Pi_\infty(T)$. The rest of this Corollary follows from (2.1).

In the following, we shall investigate the relation of the algebras $\mathcal{Q}_\theta(T)$ with various mixing concepts. But first, the concept of intermixing will be recalled.

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That T is strong mixing implies T is intermixing is clear. Friedmann and Ornestein [ ] has shown that the converse is false. If T is intermixing and $0 < m(A) < 1$, $A \in \mathcal{B}$, then

$$\liminf_{n\to\infty} m(T^nA \cap (SA)^c) > 0 \text{ for any transformation } S \in W \text{ and so,}$$

$A \notin \mathcal{Q}_\theta(T)$ for any sequence θ. Thus $\mathcal{Q}(\theta) = \mathcal{N}$. If T is ergodic with discrete spectrum then there exist a sequence θ such that $\mathcal{Q}_\theta(T) = \mathcal{B}$ and therefore $\mathcal{Q}(T) = \mathcal{N}$ implies T is weak mixing and the converse is false by examples of § 6 .

Pinsker [21] has shown that if $T\xi = \xi$ and $\Pi(T_\xi) = \nu$ then $\Pi(T)$ and ξ are independent partitions (two partitions ξ and η are independent if $m(A \cap B) = m(A)\, m(B)$ for any measurable ξ-set A and any measure η-set B see [25]). The corresponding results for the partitions $\alpha'_\theta(T)$ are shown in [28]. We shall show the corresponding result for the partitions $\alpha_\theta(T)$.

Definition 4.6.

Two Borel measures (or types of measures) on the unit circle K will be called singular modulo $\{1\}$ if their restrictions to $K-\{1\}$ are singular.

Theorem 4.7.

If $\mathcal{H}$ is a U_T-invariant subspace of $L^2(\mathcal{B})$ with $L^2(\mathcal{A}_\theta(T)) \cap \mathcal{H} = \{0\}$ or the constants, then the maximal spectral types of $U_T | L^2(\mathcal{A}_\theta(T))$ and $U_T | \mathcal{H}$ are singular modulo $\{1\}$.

Proof.

Let σ be a maximal spectral type of $T_{\alpha_\theta}(T)$ and μ be a maximal spectral type $U_T | \mathcal{H}$. If σ and μ are not singular modulo $\{1\}$, then there exists a measure τ not concentrated on $\{1\}$ with $\tau \ll \mu$. By theorem (1.7) we have

$$\int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\tau \to 0.$$

Suppose $g \in \mathcal{H}$ has τ as a spectral type then g is not constant and

$$\| U_T^{\theta(i)} g - U_T^{\theta(j)} g \|^2 = \int |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\tau \to 0.$$

So, $g \in L^2(\mathcal{A}_\theta(T))$, a contradiction.

The next Corollary is the analogue of the result of Pinsker mentioned above:

Corollary 4.8.

Suppose $T\xi = \xi$ and $\alpha_\theta(T_\xi) = \nu$. Then ξ and $\alpha_\theta(T)$ are independent partitions.

Proof.

The proof goes word for word as in [28] with $\alpha_\theta(T)$ instead of $\alpha_N(T)$, however, it will be given here for completeness sake.

By Theorem (4.9), T_ξ and $T_{\alpha_\theta}(T)$ have singular types mod $\{1\}$. Let $f \in L^2(\alpha_\theta(T))$ and $g \in L^2(\xi)$ both have integral zero. Then f and g have singular spectral types and hence are orthogonal [22].

Corollary 4.9.

If $\alpha_\theta(T) = \varepsilon$ then T is disjoint from all strong mixing transformation.

Proof.

The result follows from Theorem 4.8. and Corollary 4.8.

§5. Group Extensions

Theorem (10) of [28] and its Corollaries are valid for the partitions $\alpha_\theta(T)$ as well. Moreover, the proofs of the equivalent results for $\alpha_\theta(T)$ are almost similar. In fact, the only change will be $\alpha_\theta(T)$ instead of $\alpha_N(T)$. So, only the statements of the equivalent results will be given here.

Theorem 6.1.

Let G be a compact abelian group acting as a group of measure preserving transformations of $(X, \mathcal{B}, m)$ such that $gT = Tg$. Let $\xi(G)$ denote the partitions of X into orbits of G. If $\alpha_\theta(T_{\xi(G)}) = \nu$ then $T_{\alpha_\theta}(T)$ is conjugate to a rotation on a factor group of G. (The triviality of this factor group means $\alpha_\theta(T) = \nu$ and this will occur if T is weak-mixing).

It should be stated here that Parry [18] and Thomas [27] have proved results of this nature for $\Pi(T)$.

Corollary 5.2.

Suppose T is totally ergodic and G is a finite group acting as measure preserving transformation of (X, , m) so that $gT = Tg$ for each $g \in G$. If $\alpha_\theta (T_{\xi(G)}) = \nu$, then $\alpha_\theta(T) = \nu$.

Corollary 5.3.

i) If $\alpha_\theta(T_G) = \nu$ and T_G is weak mixing the set of $\varphi \in C_o(X,G)$ (where $C_o(X,G) = \{\emptyset: x \to G \mid \emptyset$ is continuous and $\varphi(gx) = \varphi(x)$ for all $g \in G, x \in X\}$ having the property $x \to \varphi(\cdot)\,Tx$ has $\alpha_\theta = \nu$ contains a dense G_δ in $C_o(X,G)$.

ii) If $\alpha(T_G) = \nu$ the set of $\varphi \in C_o(X,G)$ having the property that $x \to \varphi(x)T(x)$ has $\alpha = \nu$ contains a dense G_δ in $C_o(X,G)$.

Consider $Z^2 = \{1,-1\}$ with the measure which gives a weight $\frac{1}{2}$ to each point. In the following, we will consider the problem of extending a transformation with $\alpha_\theta = \varepsilon$ to obtain one with the same property. Extensions by Z^2 are the only ones that will be treated here:

Theorem 5.4.

Let $(Y, \mathcal{C}, \mu)$ be a Lebesgue space and let $X = Y \times Z^2$. Define $T: X \to X$ by $T(y,z) = (Sy, \emptyset(y)z)$ where $S: Y \to Y$ is measure preserving and $\emptyset: Y \to Z^2$ is measurable. If $\mathcal{Q}_\theta(S) = \mathcal{E}$ then

$$\mathcal{Q}_\theta(T) = \mathcal{E} \iff \int |\varphi_2(y) - \varphi_j(y)|^2 \, d\mu(y) \to 0,$$

where $\varphi_i(y) = \varphi(S^{\theta(i)}y^{-1}) \;\ldots\ldots\; \varphi(y)$, $i = 1,2, \;\ldots\ldots$.

Proof.

Suppose $\mathcal{A}_\theta(T) = \mathcal{B}$ and let $f(y,z) = z$.

Then

$$\int |\varphi_i(y) - \varphi_j|^2 \, d\mu(y) = \| U_T^{\theta(i)} f - U_T^{\theta(j)} f \|^2 \to 0 \text{ as } i,j \to \infty .$$

Conversely if

$$\int |\varphi_i(y) - \varphi_j(y)|^2 \, d\mu(y) \to 0 \text{ as } i,j \to \infty, \text{ then}$$

$\| U_T^{\theta(i)} f - U_T^{\theta(j)} f \|^2 \to 0$ and $f \in L^2 (\mathcal{A}_\theta(T))$. Hence $\mathcal{A}_\theta(T) = \mathcal{B}$.

If $\int |\varphi_i(y) - \varphi_j(y)|^2 \, d\mu(y) \to 0$, then $\varphi_i(y) \to \psi(y)$ in $L^2(Y,\mu)$, where $\psi \colon Y \to Z^2$. Also, following theorem (1.6), $\mathcal{A}_\theta(S) = \mathcal{C}$ imply $S^{\theta(i)} \to S'$ weakly where S' is an invertible measure preserving transformation of $(Y, \mathcal{C}, \mu)$ and $\mathcal{A}_\theta(T) = \mathcal{B}$ imply $T^{\theta(i)} \to T'$ weakly where T' is an invertible measure preserving transformation of $(X, \mathcal{B}, m)$. The following theorem will relate T' with S' and ψ.

Theorem 5.5

Let $(Y, \mathcal{C}, \mu)$ be a Lebesgue space and let $X = Y \times Z^2$. Define $T \colon X \to X$ by $T(y,Z) = (Sy, \varphi(y)Z)$ where $S \colon Y \to Y$ is measure preserving and $\varphi \colon Y \to Z^2$ is measurable. If $\mathcal{A}_\theta(T) = \mathcal{C}$ then $S^{\theta(i)} \to S'$ weakly where $S \in W(y)$ and $\mathcal{A}_\theta(T) = \mathcal{B}$ implies $T^{\theta(i)} \to T', T' \in W(X)$. Then T' is given by $T'(y,z) = (S'y, \psi(y)Z)$ where $\psi(y)$ is the L^2-limit of (φ_i), $\varphi_i = \varphi(S^{\theta(i)-1}y) \ldots \varphi(y)$.

Proof.

If $\mathcal{A}_\theta(S) = \mathcal{C}$ then $S^{\theta(i)} \to S'$, where $S' \in W(Y)$, according to Theorem (1.6). Also $\mathcal{A}_\theta(T) = \mathcal{B}$ implies $T^{\theta(i)} \to T'$, where $T' \in W(X)$, according to the same theorem. But, $\mathcal{A}_\theta(T) = \mathcal{B}$ implies

$\int |\varphi_i(y) - \varphi_j|^2 d\mu(y) \to 0$ by Theorem (6.4), then and this readily implies that $\varphi_i \to \psi$ in $L^2(Y,\mu)$ since $L^2(Y,\mu)$ is complete; $\psi: Y \to Z^2$ is measurable. It remains to prove that T' is given by

$T'(y,Z) = (S'y, \psi(y)Z)$. This will follow if $\| U_T^{\theta(i)}f - U_{T'}f\|^2 \to 0$ for all $f \in L^2( )$. And this will follow if the function $g(y,Z) = Z$ satisfy that condition. But

$$\| U_T^{\theta(i)}g - U_{T'}g\|^2 = \int |\varphi_i(y) - \psi(y)|^2 d\mu(y) \to 0.$$

Therefore

$$\| U_T^{\theta(i)}f - U_{T'}f\| \to 0 \text{ for all } f \in L^2(X, \mathcal{B}).$$

§6. Gaussian processes and the σ-algebras $\mathcal{C}_\theta$.

The last chapter has been devoted for the development of the theory of Gaussian processes. However, this section will be started by a brief account; at least, to retain the notations of [28].

Let $X = \prod R_n$ be the certizian product of $\mathbb{Z}$ copies of the reals i.e. $R_n = R$, $n \in \mathbb{Z}$. Let $\mathcal{C}$, be the product σ-algebra generated by the Borel sets of R. Then an element of X is a double infinite sequence $x = (x_n)_{n\in\mathbb{Z}}$. Let P_n be the co-ordinate function where $P_n(x) = x_n$. P_n is a real valued function on X. As it has been shown (cf Ch.3), a Gauss measure m can be assigned to $(X, \mathcal{C})$ by requiring that $(P_n)_{n\in\mathbb{Z}}$ be a stationary Gaussian process with covariance kernel R(n) where

$R(n) = \int \lambda^n d\mu(\lambda)$, μ is a finite measure on the unit circle $K = \{z \in \mathbb{C} \mid |z| = 1\}$ symmetric with respect to the real axis. μ is called the covariance measure of the process. Let $\mathcal{B}$ denote the completion of $\mathcal{C}$ with respect to m: It has been shown also (cf Ch.3) that $(P_n)_{n\in\mathbb{Z}}$ generates an invertible measure preserving transformation T of $(X, \mathcal{B}, m)$ and is defined by

$P_n(Tx) = P_{n-1}(x)$, and that every symmetric finite stationary Gaussian process. The transformation T will be denoted T_μ and is called the Gaussian shift based on μ.

To relate Gaussian processes to the σ-algebras $\mathcal{A}_\theta$, here is the following theorem

Theorem 6.1.

If T_μ is the Gaussian shift based on the covariance measure μ then

$$\mathcal{A}_\theta(T_\mu) = \mathcal{B} \Leftrightarrow \int_K |\lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\mu \to 0 \quad *$$

Proof

Let $\mathcal{A}_\theta(T_\mu) = \mathcal{B}$. Then by theorem (1.6) it follows that $\| U_{T_\mu}^{\theta(i)} f - U_{T_\mu}^{\theta(j)} f \|^2 \to 0$, $f \in L^2(m)$. If $f = P_1$, then

$$\| U_T^{\theta(i)} P_1 - U_T^{\theta(j)} P_1 \|^2 = \int | \lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\mu \to 0.$$

Conversely, let

$$\int | \lambda^{\theta(i)} - \lambda^{\theta(j)}|^2 d\mu \to 0, \text{ then}$$

$\| U_{T_\mu}^{\theta(i)} P_1 - U_{T_\mu}^{\theta(j)} P_1 \| \to 0$ i.e. $P_1 \in L^2(\mathcal{A}_\theta(T_\mu)$ by theorem (1.5) and hence $P_n \in L^2(\mathcal{A}_\theta(T_\mu)$ for each n and this directly implies that $L^2(\mathcal{A}_\theta(T_\mu)) = L^2(\mathcal{B})$.

Consider the space $L^2(K, \mathcal{F}, \mu)$ where $K = \{z \in \mathbb{C} \mid |z| = 1\}$ $\mathcal{F}$ is the Borel σ-algebra of K and μ is a covariance measure for a Gaussian shift (i.e. μ is a positive finite Borel measure on K). The right hand side of (*) above is equivalent to $\int |\lambda^{\theta(i)} - g(\lambda)|^2 d\mu \to 0$, where g is a limit point of the set $(\lambda^n)_{n \in \mathbb{Z}}$ in $L^2(K,\mu)$ It will turn out, after the following lemma, that all such g's have modulus 1.

Lemma 6.2.

If $\lambda^{\theta(i)} \to g$ in $L^2(K,\mathcal{F},\mu)$ where $|\lambda| = 1$ and θ is a sequence of integers then $|g| = 1$ a.e. .

Proof.

If $\lambda^{\theta(i)} \to g$ in $L^2(K, \mathcal{F},\mu)$ then $\| \lambda^{\theta(i)} - g \|^2 \to 0$

or

$$\int_K |\lambda^{\theta(i)} - g(\lambda)|^2 d\mu(\lambda) \to 0. \text{ But}$$

$$\int_K \Big| |\lambda^{\theta(i)}| - |g(\lambda)| \Big|^2 d\mu \le \int_K |\lambda^{\theta(i)} - g(\lambda)| d\mu \to 0$$

i.e. $\| 1 - |g| \|^2 \to 0$ which implies that $|g| = 1$ a.e. .

Recall (cf.Ch.3,§3,§4), that if T_μ is a Gauss shift then the induced unitary operator U_{T_μ} in $L^2(X, \mathcal{B},m)$ has the following representation $U_{T_\mu} \simeq \exp. \odot U = \sum_{n=0}^{\infty} U^{n\odot}$ in

$$L^2(X, \mathcal{B},m) \simeq \exp \odot H = \sum_{n=0}^{\infty} H^{n\odot} = \mathbb{C} \oplus H \oplus H^{2\odot} \oplus \dots\dots\dots$$

where H is a Gauss subspace and $U = U_T|H$ is cyclic in H. If $\mathcal{Q}_\theta(T_\mu) = \mathcal{B}$ then $T_\mu^{\theta(i)} \to \mathcal{J}$ weakly (cf.theorem 1.6).

The following theorem will establish that S is a generalized Gaussian shift in the sense of Kakutani [22 Theorem 2]. The theorem says that if V is a unitary operator of a fundamental Gauss subspace, of $L^2(X,\mathcal{B},m)$ onto itself, $(X,\mathcal{B},m)$ is a Lebsgue space, then its multiplicative extension to $L^2(X,\mathcal{B},m)$ is induced by a measure preserving transformation which we call generalized Gaussian shift. Certainly if V is cyclic then the transformation is a Gaussian shift.

Theorem 6.3.

Let T_μ be a Gaussian shift based on a covariance measure μ. If $\mathcal{A}_\theta(T_\mu) = \mathcal{B}$ for some sequence θ of integers then $T_\mu^{\theta(i)} \to S$ weakly and S is a generalized Gaussian shift.

Proof.

Since $U_{T_\mu}^{\theta(i)} \to U_S$, where S is an invertible measure preserving transformation in L^2 (X, ,m) and

$$U_{T_\mu} \simeq \sum_{n\geq 0} U^{n^\circ} \quad \text{we have}$$

$$(U_{T_\mu})^{\theta(i)} \simeq \sum_{n\geq 0} (U^{n\circ})^{\theta(i)}. \quad \text{But}$$

$$(U^{n^\circ})^{\theta(i)} = (U^{\theta(i)})^{n^\circ}. \quad \text{Therefore}$$

$$(U_{T_\mu})^{\theta(i)} \simeq \sum_{n\geq 0} (U^{\theta(i)})^{n^\circ}.$$ By taking the limits as $i \to \infty$.

$U_S \simeq \sum_{n\geq 0} V^{n\circ}$ where V is a unitary operator in H. Hence S is a generalized Gaussian shift.

Let $\mathcal{G}_\mu$ be the set of limit points of $(\lambda^n)_{n\in Z}$ in $L^2(K,\mu)$ where μ is a covariance measure for a Gaussian shift T_μ in $L^2(X,\mathcal{B},m)$. Theorem 6.3 established a relation between S the weak limit of $T_\mu{}^{\theta(i)}$, for some sequence θ, in W and g the limit of $\lambda^{\theta(i)}$ in $L^2(K,\mu)$. It will be proved that $\mathcal{G}_\mu$ is a group under pointwise multiplication and this relation is in fact a homomorphism between $\mathcal{G}_\mu$ and $G(T_\mu)$ the T-group of T_μ.

Lemma 6.4.

$\mathcal{G}$ is an abelian group under pointwise multiplication.

Proof.

$\mathcal{G}$ is closed under pointwise multiplication i.e. $g_1, g_2 \in$ implies $g_1 g_2 \in \mathcal{G}$. For, let $\lambda^{\theta(i)} \to g_1$, $\lambda^{\theta(i)} \to g_2$ in $L^2(K,\mu)$ for some sequence θ, θ' of integers, then

$$\lambda^{\theta(i)+\theta'(i)} \to g_1 g_2 \text{ in } L^2(K,\mu) \text{ which follows from}$$

$$\| \lambda^{\theta(i)+\theta'(i)} - g_1 g_2 \| = \| \lambda^{\theta(i)+\theta(i)} - g_1 \lambda_1^{\theta'(i)} + g_1 \lambda^{\theta'(i)} - g_1 g_2 \|$$

$$< \| \lambda^{\theta(i)} - g_1 \| + \| \; \| g_1 \| \; \| \lambda^{\theta'(i)} - g_2 \| \to 0.$$

Also $g \in \mathcal{G}$ implies that g is invertible Since if $\lambda^{\theta(i)} \to g$ then $\lambda^{-\theta(i)} \to f$ and $fg = 1$ which follows from the following

$$\| fg - 1 \| = \| fg - \lambda^{\theta(i)} f + \lambda^{\theta(i)} f - 1 \|$$

$$\leq \| f \| \; \| g - \lambda^{\theta(i)} \| + \| f - \lambda^{-\theta(i)} \| \to 0$$

Therefore $\mathcal{G}$ is an abelian group

It is obvious now the $G(T_\mu)$ is homomorphic to $\mathcal{G}$.

Theorem (14) of [28] gives an example of a weak mixing transformation, which is not strong mixing or intermixing, with $\mathcal{Q}'_\theta(T) = \mathcal{B}$. This example is a Gaussian shift based on a covaria ce measure concentrated on $D \cup D^{-1}$, where D is a Kronecker subset of K(i.e. a set D on which every continuous function supported on D and have absolute value one, can be uniformly approximated by powers of λ, $\lambda \in K = \{z \in \mathbb{C} \mid |z| = 1\}$). Let $C(D,K)$ be the set of all continuous function with support on D and have absolute value one. The following theorem will relate function in $C(D,K)$ with the algebras $\mathcal{Q}_\theta(T)$.

Theorem 6.5.

Let μ be a continuous symmetric finite measure concentrated on $D \cup D^{-1}$ where D is a Kronecker subset of K. Let T be the shift on the Gaussian process determined by μ. Then T is weak mixing and $\mathcal{Q}_\theta(T) = \mathcal{B}$ for some sequence θ (and hence is not strong mixing or intermixing)

Proof.

Let $g \in C(D,K)$, then

$$\int_{D \cup D^{-1}} |\lambda^n - g(\lambda)|^2 d\mu(\lambda) \leq 2 \int |\lambda^n - g(\lambda)|^2 d\mu(\lambda)$$

Let $\varepsilon_i \to 0$ and for each i choose $\theta(i) \in Z$ with $\sup_{\lambda \in D} |g(\lambda) - \lambda^{\theta(i)}| < \varepsilon_i$. This can be done since D is a Kronecker set and $g \in C(D,K)$

Thus $\int_D |\lambda^{\theta(i)} - g(\lambda)|^2 d\mu(\lambda) \leq \varepsilon_i^2 \mu(D) \to 0$. Therefore $\mathcal{Q}_\theta(T) = \mathcal{B}$ by theorem (6.1) with $\theta = \{\theta(i)\}$. Finally, it has been proved in [17] that such a transformation is weak mixing. The fact that it is not strong mixing or intermixing follows from the property $\mathcal{Q}_\theta(T) = \mathcal{B}$.

Remark.

Since T_μ, μ is a continuous symmetric measure concentrated on $D \cup D^{-1}$, D is a Kronecker set, is a Gaussian shift, then $G(T_\mu)$ is homomorphic to $\mathcal{G}_\mu$. But $\mathcal{G}_\mu$ is this case $C(D,K)$. Moreover, the powers of λ, i.e. the set $\{\lambda^n, n \in Z\}$, is uniformly dense in $C(D,K)$. Hence the powers of T_μ are dense in $G(T_\mu)$. In other words for every sequence of integer θ there exist an S in $G(T_\mu)$ such that, $T_\mu{}^{\theta(i)} \to S$ weakly.

The final part of this section will be about the equivalent of theorem (15) in [28], for the transformations with $\mathcal{A}_\theta(T) = \mathcal{B}$. This theorem gives a link between a transformation S, with non-discrete spectrum and $\alpha_\theta(S) = \varepsilon$, and a Gaussian shift T with $\alpha'_\theta(T) = \varepsilon$. The following theorem will construct a homomorphism between the group $G(S)$ and the group $G(T)$.

<u>Theorem 6.6</u>.

If S is an invertible measure preserving transformation which is weak mixing, then there exists a weak mixing Gaussian shift T such that G(S) is epimorphic to G(T)

<u>Proof</u>.

Let $S' \in G(S)$ with $S^{\theta(i)} \to S' = S'_\theta$ weakly then $\mathcal{A}_\theta(S) = \mathcal{B}$. Let μ_S denote a maximal spectral type of S. μ_S can be chosen symmetric with respect to the real axis. If μ is its continuous part, then μ is not trivial and symmetric by assumption. Since $\mu \ll \mu_S$, then

$$\int |\lambda^{\theta(i)} - \lambda^{\theta(i)}|^2 d\mu(\lambda) \to 0 \text{ (cf. proof of theorem (17))}$$

Let $T = T_\mu$ be the Gaussian shift based on the covariance measure μ, then $\mathcal{A}_\theta(T) = \mathcal{B}$ by theorem (6.1), and T is weak mixing since μ is a continuous measure [15]. But $\mathcal{A}_\theta(T) = \mathcal{B} \Rightarrow T_\mu^{\theta(i)} \to T' = T'_\theta$ weakly and $T' \in G(T_\mu)$ by definition of $G(T_\mu)$. Thus $T = T_\mu$ is the required Gaussian shift and the map h which take powers of S to powers of T and S'_θ to T'_θ is a homomorphism. That h is a homomorphism follows from the fact that S^{n-m} goes to T^{n-m}, $n,m \in Z$ and $S'_{\theta-\theta'}$ goes to $T'_{\theta-\theta'}$. It is clear that h is surjective.

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